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Linear differential equations in exponential extensions

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Abstract

We present an algorithm for computing rational solutions of linear differential equations with coefficients in exponential extensions of monomial extensions of a base field. We focus on the system of generators describing the extension and show why some of the generators sets are more "suitable" than others. These results partially improve and generalize the method presented by Singer (J. Symbolic Comput. 11 (1991) 251) for finding Liouvillian solutions of linear differential equations with coefficients in Liouvillian extensions of $C(x)$.

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1. Introduction

In the setting of linear differential equations, one is often interested in *rational* solutions, i.e., solutions in the field that defines the coefficients of the equation. This problem is the starting point of many other algorithms such as the factorization of linear differential operators or the computation of Liouvillian solutions of linear differential equations (see van der Put and Singer (2003) and references).

In Singer (1991), Singer presented a method for finding rational, and then Liouvillian solutions of linear differential equations with coefficients in almost all Liouvillian extensions of $C(x)$ (see Singer (1991, Theorem 4.2)). This method although effective was not very efficient. In this article, we focus on the exponential extensions. We first outline the method presented in Singer (1991) for computing the rational solutions of linear

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differential equations, and then present improvements provided by a suitable choice of the set of the exponential elements defining the extension.

Let (K, D) be a differential field. We say that θ is *exponential over* K if it is transcendental over K such that $\frac{D\theta}{\theta}$ is in K. Furthermore, we assume that the constant field is not extended, i.e., if $\mathrm{Const}(K) := \{a \in K \mid Da = 0\}$ then $\mathrm{Const}(K(\theta)) = \mathrm{Const}(K)$. Consider a monic linear differential operator

$$L = \sum_{i=0}^{n} A_i D^i \qquad \text{where } A_i \text{ is in } K(\theta) \text{ and } A_n = 1.$$

We now detail the method given in Singer (1991) for computing the rational solutions of $L(y) = 0$. Note that for the sake of simplicity we present the method in the case of a homogeneous linear differential equation, but the same arguments hold for the inhomogeneous case.

1. Compute the normal part of the denominator.

   The *normal* part of a polynomial is the part whose irreducible factors are prime with respect to their derivative (see Bronstein (1997, Definition 3.4.2) for example).

   In Singer (1991), the method proposed uses p-adic expansions to prove that the polynomials appearing in the denominator of a solution also appear in the denominators of the coefficients A_i. One then computes an indicial equation giving a bound for the valuation of this polynomial in the denominator of a solution.

   An alternative method to the p-adic expansions is proposed in Bronstein (1992), avoiding the complete factorization of the denominators of the coefficients.

   In both cases, a change of variable leads to compute Laurent polynomial solution of a linear differential equation.
2. Find a bound for the degree and the valuation of Laurent polynomial solutions.

   Let

   $$Y = \sum_{i=\delta}^{\gamma} y_i \theta^i \qquad \text{with } \begin{cases} \delta \text{ and } \gamma \text{ in } \mathbb{Z} \text{ where } \delta \leq \gamma, \\ y_i \text{ in } K \text{ and } y_\delta \neq 0, y_\gamma \neq 0 \end{cases}$$

   be such that $L(Y) = 0$ where

   $$L = \sum_{i=\nu}^{\mu} \theta^i L_i \qquad \text{with } \begin{cases} \nu \text{ and } \mu \text{ in } \mathbb{Z} \text{ where } \nu \leq \mu, \\ L_i \text{ in } K[D] \text{ and } L_\nu \neq 0, L_\mu \neq 0. \end{cases}$$

   If one writes $L(Y)$ with respect to the powers of θ, one observes that

   $$L(Y) = 0 \text{ implies that } L_\nu(y_\delta \theta^\delta) = 0 \qquad \text{and} \qquad L_\mu(y_\gamma \theta^\gamma) = 0.$$

   Then one computes the exponential solutions of L_ν and L_μ, i.e., the u and v in K such that $L_\nu(\mathrm{e}^{\int u}) = 0$ and $L_\mu(\mathrm{e}^{\int v}) = 0$. Of course, in order to do that, it is assumed that one can compute the exponential solution of linear differential equations with coefficients in K. An algorithm for computing the exponential solutions is also presented in Singer (1991). Then, given an exponential solution $\mathrm{e}^{\int w}$, one needs to decide whether $\mathrm{e}^{\int w} = f\theta^\beta$ for some f in K and β in $\mathbb{Z}$. This is equivalent to considering solutions of the equation $y' + wy = 0$ in $K(\theta)$. Using Singer (1991, Lemma 3.5) (see also Risch's work in Risch (1969, 1970) or Rothstein and Caviness (1979), Singer et al. (1985)), one can find an answer to the question in almost any Liouvillian extension of $C(x)$. This gives bounds for the degree γ and the valuation δ of a solution Y.

3. Compute the coefficients of Laurent polynomial solutions.

   Writing $L(Y)$ with respect to the powers of θ, one has a linear differential system with coefficients in K in the form $AY = B$. Using non-commutative linear algebra (see Poole (1960) for example), one finds matrices U and V in $K[D]$ such that U has a left inverse, V has a right inverse, and $UAV = C$ is diagonal. So Y is a solution of $AY = B$ if and only if $W = V^{-1}Y$ is a solution of $CW = UB$. This system is equivalent to $n+1$ linear differential equations with coefficients in K, where n is the order of the linear differential equation L.

Singer's method is algorithmic but some steps are not easy, such as the computation of the exponential solutions and the identification of the suitable ones in the second step, or the diagonalization of the matrix in the third step.

In Bronstein and Fredet (1999), several improvements are obtained for the last two steps in the case of extensions of the form $C(x, \mathrm{e}^{\int f(x)\mathrm{d}x})$ and in Fredet (2000) this method is adapted to extensions of $C(x)$ generated by iterated logarithms and exponentials. In this article, we propose improvements for these steps when considering exponential extensions with a more general form.

First, in Bronstein and Fredet (1999), concerning the third step, we had pointed out that the system has a special form: if $L(Y) = 0$ for $Y = y_\delta\theta^\delta + \cdots + y_\gamma\theta^\gamma$ then $M\overrightarrow{Y} = 0$ where M is a matrix with linear differential operators as coefficients, and has the form

$$M = \begin{pmatrix} * & 0 & 0 & \cdots & \\ * & * & 0 & \cdots & \\ \vdots & & \ddots & & \\ * & * & \cdots & * & 0 \\ * & * & \cdots & * & * \\ \vdots & & & & \vdots \\ * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ 0 & 0 & \ddots & * & * \\ \vdots & & & \ddots & \\ 0 & 0 & \cdots & 0 & * \end{pmatrix} \quad \text{and} \quad \overrightarrow{Y} = \begin{pmatrix} y_\delta \\ y_{\delta+1} \\ \vdots \\ y_{\gamma-1} \\ y_\gamma \end{pmatrix}.$$

We used this special form to avoid writing the system down. In this article, we link this approach to the recurrence relation used in Abramov et al. (1995) to solve linear differential equations with coefficients in $C(x)$, and generalize it to a larger class of exponential extensions.

In the second step, we show how to find a suitable set of generators defining the exponential extension so that, focusing directly on solutions of the form $f\theta^\beta$ for f in

K and β in $\mathbb{Z}$, a simple algorithm gives us β. Then we have directly the bounds on the degree and the valuation of Laurent polynomial solutions of L. Let us illustrate this with an example:

Example 1. Consider the following linear differential equation:

$$L(y) := (1 + 6x)y'' + (-60x^2 - 13 - 52x)y' + (96x^3 + 28 + 104x + 208x^2)y = 0.$$

Assume we are interested in the solutions in $C(x, \exp(x^2), \exp(x^2 + x))$. We use the notation $\theta_1 = \exp(x^2)$ and $\theta_2 = \exp(x^2 + x)$. If there exists Y in $C(x)[\theta_1, \theta_2, \theta_1^{-1}, \theta_2^{-1}]$ such that $L(Y) = 0$ then there are solutions of the form $Y = f\theta_1^{\gamma_1}\theta_2^{\gamma_2}$ with f in $C(x)$ and γ_1, γ_2 in $\mathbb{Z}$ (see Rosenlicht (1975, Theorem 1)).

Therefore, the following equalities hold:

$$\begin{aligned} Y' &= (f' + (2x)\gamma_1 f + (2x + 1)\gamma_2 f)\theta_1^{\gamma_1}\theta_2^{\gamma_2}, \\ &= (2x(\gamma_1 + \gamma_2) f + \gamma_2 f + f')\theta_1^{\gamma_1}\theta_2^{\gamma_2}. \\ Y'' &= (4x^2(\gamma_1 + \gamma_2)^2 + \cdots)\theta_1^{\gamma_1}\theta_2^{\gamma_2}. \end{aligned}$$

Expanding f as a series at infinity ($f = cx^\alpha + \cdots x^{\alpha-1} + \cdots$ with $c \neq 0$ and α in $\mathbb{Z}$), we conclude that the leading term of Y' is $2(\gamma_1 + \gamma_2)c$ and that the one of Y'' is $4(\gamma_1 + \gamma_2)^2c$. Therefore $L(Y) = 0$ implies that

$$c(6 \times 4(\gamma_1 + \gamma_2)^2 - 60 \times 2(\gamma_1 + \gamma_2) + 96) = 0.$$

Since $c \neq 0$, we have a finite set of choices for $\gamma_1 + \gamma_2$, but neither for γ_1 nor γ_2. If we consider $C(x, \exp(x^2), \exp(x))$ as the base field, which is differentially isomorphic to $C(x, \exp(x^2 + x), \exp(x^2))$, the previous computation provides us with a finite set of choices for the exponents of $\exp(x^2)$—here $\{1, 4\}$. After a change of variable, we proceed in the same way and find a finite set of choices for the exponents of $\exp(x)$. We have then a finite set of choices for (γ_1, γ_2). Changing the variables reduces the problem to finding rational solutions of linear differential equations with coefficients in $C(x)$.

The previous example shows us that if the system of generators is suitably chosen it is possible to find bounds directly by using expansions at infinity. We have also used our definition of a suitable system of generators in Fredet (2001) to compute rational solutions of linear differential systems with coefficients in exponential extensions.

This paper is organized as follows. In the next section we present the improvements of Singer's method in *flat* exponential extensions of a suitable monomial extension $k(t)$ of a base field k. Such extensions are defined by adding simultaneously several exponential variables over $k(t)$. We introduce the notion of a *well-defined* extension which formalizes our notion of a "suitable" set of generators. We prove that in such extensions the computation of bounds on the degree and the valuation of Laurent polynomial solutions of linear differential equations requires only expansions at infinity or p-adic expansions for some irreducible polynomial p. This allows us to consider extensions that are not Liouvillian, and extend some of the results presented in Singer (1991). In Section 3, we present an algorithm which, given a set of generators for a flat exponential extension, computes a suitable set of generators such that this extension is well defined.

2. Laurent polynomial solutions of linear differential equations

2.1. Exponential extensions over monomial extensions of a base field

In this section, we consider a differential field (k, D) where k is a field with a known $\mathbb{Q}$-basis. This implies that one can compute the integer roots of polynomials with coefficients in k.

An element t is *monomial* over k if t is transcendental over k and Dt is a polynomial in t. The extension $k(t)$ is then a *monomial extension* of k. Furthermore, we consider only monomial extensions $k(t)$ such that $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$ where $\mathrm{Const}(k) = \{a \in k \mid Da = 0\}$.

Let g in $k(t)$ be such that $g = N/M$ for N and M in $k[t]$. Then we define $\deg_t(g) = \deg_t(N) - \deg_t(M)$.

We will propose an algorithm for computing Laurent polynomial solutions of linear differential equations with coefficients in exponential extensions of $k(t)$. Using Singer (1991) or Bronstein (1992), this will give us an algorithm for computing the rational solutions of such equations. So it is natural to assume that one can solve this problem in $k(t)$:

Definition 2.1. We say that we can *effectively solve parametrized linear differential equations over* $k(t)$ if given any linear ordinary differential operator $L = \sum_{i=0}^{n} a_i D^i$ with coefficients in $k(t)$ and $g_1, \ldots, g_m$ in $k(t)$, one can effectively find $y_1, \ldots, y_r$ in $k(t)$ and a matrix M with coefficients in $\mathrm{Const}(k)$ such that $L(y) = \sum_{j=1}^{m} c_j g_j$ for y in $k(t)$ and $c_1, \ldots, c_m$ in $\mathrm{Const}(k)$ if and only if $y = \sum_{k=1}^{r} h_k y_k$ where $h_1, \ldots, h_r$ are in $\mathrm{Const}(k)$ and $M(h_1, \ldots, h_r, c_1, \ldots, c_m)^{\mathrm{T}} = 0$.

In all that follows $k(t)$ denotes a monomial extension of k such that $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$. As explained before, we assume that we can effectively solve parametrized linear differential equations over $k(t)$. We now turn our attention to the addition of exponential elements over $k(t)$. In order to use p-adic expansions or expansions at infinity, we limit the extensions considered. The method relies on the hypothesis that, for some valuation, the orders of the logarithmic derivatives of the exponential variables are greater than the orders of the logarithmic derivatives of the functions of $k(t)$, or the leading terms are $\mathbb{Q}$-linearly independent. This hypothesis is always true in exponential extensions of $C(x)$ but not in exponential extensions of monomial extensions of a base field. Corollary 4.4.2 in Bronstein (1997) shows that for any f in $k(t)$, $\mathrm{val}_p(\frac{Df}{f}) \geq -1$ and if $\mathrm{val}_p(\frac{Df}{f}) = -1$ then the residue is $\mathrm{val}_p(f)$. Furthermore, Theorem 4.4.4 in Bronstein (1997) shows that $\deg_t(\frac{Df}{f}) \leq \deg_t(Dt) - 1$ for any f in $k(t)$. So we choose to restrict the exponential extensions considered, in the following way:

Definition 2.2. An element θ is *effectively exponential over* $k(t)$ if

1. $\frac{D\theta}{\theta}$ is in $k(t)\backslash k$,
2. for all c in $\mathbb{Q}$ and f in $k(t)^*$, $\frac{D\theta}{\theta} + c\frac{Df}{f}$ has a $\frac{1}{t}$-adic expansion containing an element with the form $u_\beta t^\beta$ where β is an integer such that

   (a) either β is greater than or equal to $\max(1, \deg_t(Dt))$,
   (b) or $\beta < 0$.

A differential extension F of $k(t)$ is an *effectively exponential extension over* $k(t)$ if $F = k(t,\theta)$ where $\theta \in F^*$ is effectively exponential over $k(t)$.

In the previous definition, if $\beta \geq \max(1, \deg_t(Dt))$, this will be true for all c and f since $\deg_t(\frac{Df}{f}) \leq \deg_t(Dt) - 1$. So, if $\beta < 0$ there exists $p \in k[t]$ such that either $\mathrm{val}_p(\frac{D\theta}{\theta}) < -1$, or $\mathrm{val}_p(\frac{D\theta}{\theta}) = -1$ and the leading term in the p-adic expansion is $\mathbb{Q}$-linearly independent of $(Dp \bmod p)$. To see this, note that $\beta < 0$ implies that, for any f and c as above, we have that $\frac{D\theta}{\theta} + c\frac{Df}{f}$ is not polynomial in t. If some irreducible factor p of the denominator of $\frac{D\theta}{\theta}$ appears to a power larger than 1, we are done. Otherwise, we can write $\frac{D\theta}{\theta} = F + \sum \frac{q_i}{p_i}$ where q_i, p_i are in $k[t]$, irreducible and such that $\deg_t(q_i) < \deg_t(p_i)$ and F is a polynomial such that $\deg_t(F) < \max(1, \deg_t(Dt))$. We claim that some q_i is $\mathbb{Q}$-linearly independent of $Dp_i \bmod p_i$. If not, we write $q_i = \frac{c_i}{N}Dp_i$ with $c_i, N \in \mathbb{Z}$. If we let $f = \prod q_i^{c_i}$ then $\frac{D\theta}{\theta} - \frac{1}{N}\frac{df}{f}$ is a polynomial with degree less than $\max(1, \deg_t(Dt))$, which is a contradiction.

Remark 1. Definition 2.2 implies that there is always an expansion that allows us to distinguish one term of $\frac{D\theta}{\theta}$ from any term of the logarithmic derivative of any function f in $k(t)$.

Example 2. Let t be transcendental over $C(x)$ and such that $Dt = t^2 + x$.

Then $C(x, t, \mathrm{e}^{\int t})$ is not effectively exponential over $C(x,t)$ because the order of $\frac{D\mathrm{e}^{\int t}}{\mathrm{e}^{\int t}} = t$ is 1, which is not greater than or equal to $\max(1, \deg_t(Dt)) = 2$. Therefore condition (2) is false for $c = 0$.

But $C(x, t, \mathrm{e}^{\int t^3})$ is effectively exponential over $C(x,t)$ because conditions (1) and (2a) are true.

Remark 2. The definition implies that for all c in $\mathbb{Q}$, for all f in $k(t)^*$, we have $\theta \neq f^c$. This means that $n\frac{D\theta}{\theta} \neq \frac{Dv}{v}$ for all integer $n \neq 0$, for all $v \in k(t)^*$ and θ is exponential over $k(t)$. As a consequence (see Bronstein (1997, Theorem 5.1.2)), θ is transcendental over $k(t)$ and $\mathrm{Const}(k(t,\theta)) = \mathrm{Const}(k(t)) = \mathrm{Const}(k)$.

Let us now consider the addition of several exponential variables:

Definition 2.3. A field E is a *flat effectively exponential extension of* $k(t)$ if there are $\theta_1, \ldots, \theta_l$ such that $E = k(t, \theta_1, \ldots, \theta_l)$ and for all c_i in $\mathbb{Q}$ not all zero, $\prod_{i=1}^{l} \theta_i^{c_i}$ is effectively exponential over $k(t)$.

Proposition 2.1. *If $E = k(t, \theta_1, \ldots, \theta_l)$ is a flat effectively exponential extension of $k(t)$ then the θ_i's are algebraically independent over $k(t)$.*

Proof. This follows from Theorem 1 in Rosenlicht (1975): if $\sum f_i$ is algebraic over $k(t)$ with $f_i = \prod_j \theta_j^{c_{i,j}}$ exponential over $k(t)$ such that $\frac{f_i}{f_j} \notin k(t)$ for $i \neq j$ then each f_i is algebraic over $k(t)$. This is false by hypothesis. $\square$

Remark 3. If the θ_i's are effectively exponential and algebraically independent, we cannot conclude that the extension is flat effectively exponential: assume that we consider the same field as in Example 2, say $K = C(x,t)$ such that $Dt = t^2 + x$. Then consider θ_1 and θ_2 such that $\frac{D\theta_1}{\theta_1} = t^3 + t$ and $\frac{D\theta_2}{\theta_2} = t^3$ are both effectively exponential and algebraically independent over K but θ_1/θ_2 is not effectively exponential and so the extension $K(\theta_1, \theta_2)$ is not a flat effectively exponential extension of K.

Notation. We denote $\{1, \ldots, l\}$ by $\mathcal{L}$, $k(t, \theta_1, \ldots, \theta_l)$ by $k(t, \theta_{\mathcal{L}})$, $k(t)[\theta_1, \ldots, \theta_l]$ by $k(t)[\theta_{\mathcal{L}}]$, and $k(t)[\theta_1, \ldots, \theta_l, \theta_1^{-1}, \ldots, \theta_l^{-1}]$ by $k(t)[\theta_{\mathcal{L}}, \theta_{\mathcal{L}}^{-1}]$.

Let us focus on the system of generators defining flat effectively exponential extensions:

Definition 2.4. Let $k(t, \theta_{\mathcal{L}})$ denote a flat effectively exponential extension of $k(t)$ and write $g_i = \frac{D\theta_i}{\theta_i}$. For any subset $\mathcal{N}$ of $\mathcal{L} = \{1, \ldots, l\}$ we define

$$\mathcal{Q}^{\mathcal{N}} = \left\{ j \in \mathcal{N} \mid \deg_t(g_j) = \max\left(\deg_t(Dt), 1, \max_{k \in \mathcal{N}}(\deg_t(g_k))\right)\right\},$$

and for any irreducible polynomial $p \in k[t]$, we let

$$\mathcal{Q}_p^{\mathcal{N}} = \left\{ j \in \mathcal{N} \mid \mathrm{val}_p(g_j) = \min\left(-1, \min_{k \in \mathcal{N}}(\mathrm{val}_p(g_k))\right)\right\}.$$

We say that $k(t, \theta_{\mathcal{L}})$ is a *well-defined exponential extension* of $k(t)$ if for any non-empty subset $\mathcal{N} \subset \mathcal{L}$ we have

1. either: $\mathcal{Q}^{\mathcal{N}} \neq \emptyset$ and if for j in $\mathcal{Q}^{\mathcal{N}}$ we write $g_j = u_j t^{\alpha} + \cdots$ then the u_j's are $\mathbb{Q}$-linearly independent,
2. or: $\mathcal{Q}^{\mathcal{N}} = \emptyset$ and there exist an irreducible polynomial p such that $\mathcal{Q}_p^{\mathcal{N}} \neq \emptyset$. Furthermore, if we write $g_j = \frac{u_j}{p^{\alpha}} + \cdots$ for $j \in \mathcal{Q}_p^{\mathcal{N}}$ then
   (a) either: $\alpha > 1$ and the u_j's are $\mathbb{Q}$-linearly independent,
   (b) or: $\alpha = 1$ and the u_j's and (Dp mod p) are $\mathbb{Q}$-linearly independent.

Remark 4. The polynomials p such that $\mathcal{Q}_p^{\mathcal{N}}$ is not empty appear in the denominator of some g_j for j in $\mathcal{N}$. So, there are only a finite number of such polynomials p to consider, and we can compute them. We will denote by $\mathcal{P}^{\mathcal{N}}$ this set:

$$\mathcal{P}^{\mathcal{N}} = \left\{ \text{irreducible } p \in k[t] \mid \exists j \in \mathcal{N} \text{ such that } \mathrm{val}_p\left(\frac{D\theta_j}{\theta_j}\right) < 0\right\}.$$

Remark 5. A flat extension is well defined if, for some valuation, the logarithmic derivatives of the exponential variables have different orders or leading terms $\mathbb{Q}$-linearly independent, possibly also $\mathbb{Q}$-linearly independent of Dp mod p in the case of order one.

Example 3. The extension $C(x, \mathrm{e}^{\int x^3+x}, \mathrm{e}^{\int x^3+3x})$ is a flat exponential extension of $C(x)$. It is not well defined but is isomorphic to $C(x, \mathrm{e}^{\int x^3+x}, \mathrm{e}^{\int 2x})$ which is well defined.

2.2. Riccati equation

We consider a monomial extension $k(t)$ of a differential field (k, D). For u in $k(t)$, we define

$$P_0 = 1,$$
$$P_i = DP_{i-1} + uP_{i-1} \qquad \text{for } i \geq 1.$$

If we consider a linear differential operator $L = A_n D^n + A_{n-1} D^{n-1} + \cdots + A_0$ with coefficients in $k(t)$ then for any y and $u = \frac{Dy}{y}$, we have

$$L(y) = 0 \Leftrightarrow A_n P_n(u, \ldots, D^{n-1}u) + A_{n-1} P_{n-1}(u, \ldots, D^{n-2}u) + \cdots + A_0 = 0.$$

Some properties of the polynomial that we will use for the p-adic expansions have to be distinguished: a polynomial p in $k[t]$ is *special* if p divides Dp and p is *normal* if p and Dp are coprime (see Bronstein (1997, Definition 3.4.2)). We remark that an irreducible polynomial is either special or normal. Adapting Singer (1991, Lemma 2.2), we have:

Lemma 2.2. *Let u be in $k(t)$ with the following expansion: $u = u_\beta t^\beta +$ terms with lower order, with β in $\mathbb{Z}_{>0}$, u_β in k, $u_\beta \neq 0$, and write $\alpha = \deg_t(Dt)$. If $\beta \geq \max(\alpha, 1)$, then $P_i = (u_\beta)^i t^{i\beta} +$ terms with lower order.*

Proof. By induction on i. □

Lemma 2.3. *Let $p(t)$ be a polynomial in $k[t]$ normal and irreducible. Let u be in $k(t)$ with the following p-adic expansion: $u = \frac{u_\beta}{p^\beta} +$ higher order terms, with β in $\mathbb{Z}_{>0}$, u_β in $k[t]$ such that $u_\beta \neq 0$ and $\deg_t(u_\beta) < \deg_t(p)$.*

1. *If $\beta > 1$, then $P_i(u, \ldots, D^{i-1}u) = \frac{v_{i,\beta}}{p^{i\beta}} +$ higher order terms, where $v_{i,\beta} \equiv (u_\beta)^i \bmod p$.*
2. *If $\beta = 1$ and u_β is prime with $(Dp \bmod p)$, then $P_i(u, \ldots, D^{i-1}u) = \frac{v_i}{p^i} +$ higher order terms where $v_i \equiv \prod_{j=0}^{i-1}(u_1 - jDp) \bmod p$.*

Proof. By induction on i. □

Lemma 2.4. *Let $p(t)$ be a polynomial in $k[t]$ special and irreducible. Let u be in $k(t)$ with the following p-adic expansion: $u = \frac{u_\beta}{p^\beta} +$ higher order terms, with β in $\mathbb{Z}_{>0}$, u_β in $k[t]$ such that $u_\beta \neq 0$ and $\deg_t(u_\beta) < \deg_t(p)$. Then $P_i(u, \ldots, D^{i-1}u) = \frac{v_{i,\beta}}{p^{i\beta}} + \cdots$ where $v_{i,\beta} \equiv (u_\beta)^i \bmod p$.*

Proof. We note that if p is special then p divides Dp: $Dp = qp$ for q in $k[t]$. Then we proceed by induction on i. □

2.3. Bounds on the degree and valuation of Laurent polynomial solutions

We search for bounds on the degree and the valuation of Laurent polynomial solutions of linear differential equations with coefficients in well-defined exponential extensions (see Definition 2.4).

During the bounding process, we have to compute the integer solutions of some algebraic equations. The following lemma shows that the systems that arise have finitely many such solutions.

Lemma 2.5. *Let $K \subset F$ be fields and $p \in F[Z]$ be irreducible with $\deg(p) = m$. Let $a_0, \ldots, a_n$ be in $F[Z]$ and $q, u_1, \ldots, u_l$ in $F[Z]$ be linearly independent over K and such* *that* $\deg(q) < m$ *and* $\deg(u_i) < m$ *for* $i = 1, \ldots, l$. *If* p *does not divide* $\gcd(a_0, \ldots, a_n)$, *then the set*

$$S = \left\{ (\gamma_1, \ldots, \gamma_l, \alpha) \in K^{l+1} \text{ such that } \sum_{i=0}^{n} a_i \left(\sum_{j=1}^{l} \gamma_j u_j + \alpha q \right)^i \equiv 0 \bmod p \right\}$$

is finite with at most n elements.

Proof. Let $\pi : F[Z] \to F[Z]/(p)$ be the canonical projection. We consider the polynomial $Q = \sum_{i=0}^{n} \pi(a_i) T^i$ in $(F[Z]/(p))[T]$. Since p does not divide $\gcd(a_0, \ldots, a_n)$, Q is not identically 0 and has degree at most n, so let $\beta_1, \ldots, \beta_k$ with $k \leq n$ be its distinct roots in $F[Z]/(p)$, and let $p_1, \ldots, p_k \in F[Z]$ be such that $\deg(p_j) < m$ and $\pi(p_j) = \beta_j$ for each j. For any $(\gamma_1, \ldots, \gamma_l, \alpha)$ in S, we have $Q(\sum_j \gamma_j \pi(u_j) + \alpha \pi(q)) = 0$, which implies that $\pi(\gamma_1 u_1 + \cdots + \gamma_l u_l + \alpha q) = \pi(p_j)$ for some j, and hence that $\gamma_1 u_1 + \cdots + \gamma_l u_l + \alpha q = p_j$ since the degrees of $u_1, \ldots, u_l, q$ and p_j are strictly less than m. But for each j, there is at most one $(l+1)$-uplet $(\gamma_1, \ldots, \gamma_l, b) \in K^{l+1}$ such that $\gamma_1 u_1 + \cdots + \gamma_l u_l + bq = p_j$, since the difference of two such relations with distinct pairs would yield a linear dependence over K for the u_j's and q. Therefore, S has at most n elements. $\square$

There are many algorithms for computing the integer roots of such a polynomial if the field has a known $\mathbb{Q}$-basis as we assumed. For example, see Geddes et al. (1992) for computing the roots β_i and then use Gaussian elimination for computing the γ_i's.

Proposition 2.6. *Let k be a differential field with a known* $\mathbb{Q}$*-basis and* $k(t)$ *be a monomial extension of* k *such that* $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$. *Let* $K = k(t, \theta_{\mathcal{L}})$ *be a well-defined exponential extension of* $k(t)$. *Let* $L = \sum_{i=1}^{n} A_i D^i$ *be a linear differential operator with coefficients in* $k(t)$. *There exists a computable set* $\mathcal{S}$ *such that if there exists* $Y = f\theta_1^{\beta_1} \ldots \theta_l^{\beta_l}$ *obeying* $L(Y) = 0$ *with* $f \neq 0$ *in* $k(t)$ *and* β_i *in* $\mathbb{Z}$, *then* $(\beta_1, \ldots, \beta_l)$ *belongs to this set* $\mathcal{S}$.

Proof. Denote $\frac{D\theta_i}{\theta_i}$ by g_i. We proceed by induction on l. If $l = 0$, it is trivial. Assume that $l > 0$ and that there exists $Y = f\theta_1^{\beta_1} \ldots \theta_l^{\beta_l}$ where f is in K and $(\beta_1, \ldots, \beta_l)$ in $\mathbb{Z}^l$ such that $L(Y) = 0$. Then, from Definition 2.4, there are two possibilities:

1. *Case* 1. $\mathcal{Q}^{\mathcal{L}} \neq \emptyset$ where

$$\mathcal{Q}^{\mathcal{L}} = \left\{ j \in \mathcal{L} \mid \deg_t(g_j) = \max\left(\deg_t(Dt), 1, \max_{k \in \mathcal{L}}(\deg_t(g_k)) \right) \right\}.$$

   Furthermore, if for j in $\mathcal{Q}^{\mathcal{L}}$ we write $g_j = u_j t^{\alpha} + \cdots$ where $\alpha = \max_{j \in \mathcal{L}}(\deg_t(g_j))$, then the u_j's are $\mathbb{Q}$-linearly independent. Note that $\mathcal{Q}^{\mathcal{L}} \neq \emptyset$ implies $\alpha \geq \max(1, \deg_t(Dt))$.
2. *Case* 2. $\mathcal{Q}^{\mathcal{L}} = \emptyset$.

   Then there exists an irreducible polynomial p in $k[t]$ such that $\mathcal{Q}_p^{\mathcal{L}} \neq \emptyset$ where

$$\mathcal{Q}_p^{\mathcal{L}} = \left\{ j \in \mathcal{L} \mid \mathrm{val}_p(g_j) = \min\left(-1, \min_{k \in \mathcal{L}}(\mathrm{val}_p(g_k)) \right) \right\}.$$

Furthermore, if for j in $\mathcal{Q}_p^{\mathcal{L}}$, we write $g_j = \frac{u_j}{p^\alpha} + \cdots$ then

(a) either $\alpha > 1$ and the u_j's are $\mathbb{Q}$-linearly independent,
(b) or $\alpha = 1$ and the u_j's and $(Dp \bmod p)$ are $\mathbb{Q}$-linearly independent.

We will consider these possibilities:

1. *Case* 1. $\mathcal{Q}^{\mathcal{L}} \neq \emptyset$ where

$$\mathcal{Q}^{\mathcal{L}} = \left\{ j \in \mathcal{L} \mid \deg_t(g_j) = \max\left(\deg_t(Dt), 1, \max_{k\in\mathcal{L}}(\deg_t(g_k))\right)\right\}.$$

So, if for j in $\mathcal{Q}^{\mathcal{L}}$ we write $g_j = u_j t^\alpha + \cdots$ where $\alpha = \max_{j\in\mathcal{L}}(\deg_t(g_j))$, then the u_j's are $\mathbb{Q}$-linearly independent.

Theorem 4.4.4 in Bronstein (1997) shows that $\deg_t(\frac{Df}{f}) \leq \deg_t(Dt) - 1$ for any f in $k(t)$. Then if $Y = f\theta_1^{\beta_1}\ldots\theta_l^{\beta_l}$, we have $\frac{DY}{Y} = (\sum_{k\in\mathcal{Q}^{\mathcal{L}}} \beta_k u_k)t^\alpha +$ lower order terms. As the u_j's are $\mathbb{Q}$-linearly independent, the degree of $\frac{DY}{Y}$ is α. From Lemma 2.2 we have $P_i = (\sum_{k\in\mathcal{Q}^{\mathcal{L}}} \beta_k u_k)^i t^{i\alpha} +$ lower order terms. As $L(Y) = 0$ we have $\sum A_i P_i = 0$ and hence the leading term of $\sum A_i P_i$ vanishes. We write

$$A_i = a_i t^{\psi_i} + \cdots \text{ where } a_i \text{ is in } k \text{ and } \psi_i \text{ in } \mathbb{Z}.$$

This gives us

$$\sum_{i\mid\psi_i + i\alpha = \max(\psi_j + j\alpha)} a_i \left(\sum_{k\in\mathcal{Q}^{\mathcal{L}}} \beta_k u_k\right)^i = 0.$$

The $(u_i)_{i\in\mathcal{Q}^{\mathcal{L}}}$ are $\mathbb{Q}$-linearly independent and $k(t)$ has a known basis over $\mathbb{Q}$. We can compute a finite set of choices for $(\beta_i)_{i\in\mathcal{Q}^{\mathcal{L}}}$. For each choice of $(\beta_i)_{i\in\mathcal{Q}^{\mathcal{L}}}$, we let $Y = z\prod_{i\in\mathcal{Q}^{\mathcal{L}}} \theta_i^{\beta_i}$ and $\widehat{\mathcal{L}} = \mathcal{L}\backslash\mathcal{Q}^{\mathcal{L}}$. If $\widehat{\mathcal{L}} = \emptyset$, then z is in $k(t)$ and we are done. Else we compute $\widehat{L} = S_{D\to D+\sum_{i\in\mathcal{Q}^{\mathcal{L}}}\beta_i g_i}(L)$, i.e., $\widehat{L}$ is the resulting operator obtained from the operator L if D is replaced by $D + \sum_{i\in\mathcal{Q}^{\mathcal{L}}} \beta_i g_i$. Since $\widehat{L}(z) = 0$ if and only if $L(Y) = 0$, we apply the above to $k(t, \theta_{\widehat{\mathcal{L}}})$ and $\widehat{L}(z) = 0$. Note that $|\widehat{\mathcal{L}}| < |\mathcal{L}| = l$, so we can compute the corresponding $(\beta_i)_{i\in\widehat{\mathcal{L}}}$ by the induction hypothesis.

2. *Case* 2. $\mathcal{Q}^{\mathcal{L}} = \emptyset$.

Then there exists an irreducible polynomial p in $k[t]$ appearing in the denominator of some g_i for i in $\mathcal{L}$, such that $\mathcal{Q}_p^{\mathcal{L}} \neq \emptyset$ where

$$\mathcal{Q}_p^{\mathcal{L}} = \left\{ j \in \mathcal{L} \mid \mathrm{val}_p(g_j) = \min\left(-1, \min_{k\in\mathcal{L}}(\mathrm{val}_p(g_k))\right)\right\}.$$

Furthermore, if for j in $\mathcal{Q}_p^{\mathcal{L}}$, we write $g_j = \frac{u_j}{p^\alpha} + \cdots$, then

(a) either $\alpha > 1$ and the u_j's are $\mathbb{Q}$-linearly independent,
(b) or $\alpha = 1$ and the u_j's and $Dp \bmod p$ are $\mathbb{Q}$-linearly independent.

Corollary 4.4.2 in Bronstein (1997) shows that for any f in $k(t)$, $\mathrm{val}_p(\frac{Df}{f}) \geq -1$ and if $\mathrm{val}_p(\frac{Df}{f}) = -1$, then p is normal and the residue is $\mathrm{val}_p(f)$. So we deduce that if $Y = f\theta_1^{\beta_1}\dots\theta_l^{\beta_l}$, then

$$\frac{DY}{Y} = \frac{\left(\sum_{k\in\mathcal{Q}_p^{\mathcal{L}}} \beta_k u_k\right)}{p^\alpha} + \cdots \qquad \text{if } \alpha > 1 \qquad \text{and}$$

$$\frac{DY}{Y} = \frac{\left(\sum_{k\in\mathcal{Q}_p^{\mathcal{L}}} \beta_k u_k\right) + mDp}{p} + \cdots \qquad \text{if } \alpha = 1,$$

where m is the valuation of f at p.

(a) If p is normal and $\alpha > 1$, then, using Lemma 2.3, $P_i(u, \dots, D^{i-1}u) = \frac{v_{i,\alpha}}{p^{i\alpha}} +$ higher order terms, where

$$v_{i,\alpha} \equiv \left(\sum_{k\in\mathcal{Q}_p^{\mathcal{L}}} \beta_k u_k\right)^i \bmod p.$$

(b) If p is normal and $\alpha = 1$, then, using Lemma 2.3, $P_i(u, \dots, D^{i-1}u) = \frac{v_i}{p^i} +$ higher order terms, where

$$v_i \equiv \prod_{j=0}^{i-1}\left(\left(\sum_{k\in\mathcal{Q}_p^{\mathcal{L}}} \beta_k u_k\right) - (j-m)Dp\right) \bmod p.$$

(c) If p is special, then, using Lemma 2.4, $P_i(u, \dots, D^{i-1}u) = \frac{v_{i,\alpha}}{p^{i\alpha}} +$ higher order terms, where

$$v_{i,\alpha} \equiv \left(\sum_{k\in\mathcal{Q}_p^{\mathcal{L}}} \beta_k u_k\right)^i \bmod p.$$

But $L(Y) = 0$ is equivalent to $\sum A_i P_i = 0$ and then the leading term of the p-adic expansion of $\sum A_i P_i$ vanishes. We write

$$A_i = \frac{a_i}{p^{\psi_i}} + \cdots \qquad \text{where } a_i \text{ is in } k[t],\ \deg_t(a_i) < \deg_t(p) \text{ and } \psi_i \text{ is in } \mathbb{Z}.$$

Then:

(a) If p is normal and $\alpha > 1$, this gives

$$\sum_{i\,|\,\psi_i + i\alpha = \max(\psi_j + j\alpha)} a_i \left(\sum_{k\in\mathcal{Q}_p^{\mathcal{L}}} \beta_k u_k\right)^i \equiv 0 \bmod p.$$

(b) If p is normal and $\alpha = 1$, this gives

$$\sum_{i|\psi_i+i=\max(\psi_j+j)} a_i \left(\prod_{j=0}^{i-1}\left(\left(\sum_{k\in\mathcal{Q}_p^{\mathcal{L}}}\beta_k u_k\right)-(j-m)Dp\right)\right)^i \equiv 0 \bmod p.$$

(c) If p is special this gives

$$\sum_{i|\psi_i+i\alpha=\max(\psi_j+j\alpha)} a_i \left(\sum_{k\in\mathcal{Q}_p^{\mathcal{L}}}\beta_k u_k\right)^i \equiv 0 \bmod p.$$

The $(u_i)_{i\in\mathcal{Q}_p^{\mathcal{L}}}$ are $\mathbb{Q}$-linearly independent, so by Lemma 2.5 we have a finite set of choices for $(\beta_i)_{i\in\mathcal{Q}_p^{\mathcal{L}}}$. For each choice of $(\beta_i)_{i\in\mathcal{Q}_p^{\mathcal{L}}}$, we let $Y = z\prod_{i\in\mathcal{Q}_p^{\mathcal{L}}}\theta_i^{\beta_i}$ and $\widehat{\mathcal{L}} = \mathcal{L}\backslash\mathcal{Q}^{\mathcal{L}}$. If $\widehat{\mathcal{L}} = \emptyset$, then z is in $k(t)$ and we are done. Else we compute $\widehat{L}$ such that $\widehat{L}(z) = 0$ if and only if $L(Y) = 0$, i.e., $\widehat{L} = S_{D\to D+\sum_{i\in\mathcal{Q}_p^{\mathcal{L}}}\beta_i g_i}(L)$ and we apply the above to $k(t,\theta_{\widehat{\mathcal{L}}})$ and $\widehat{L}(z) = 0$. Note that $|\widehat{\mathcal{L}}| < |\mathcal{L}|$ so this occurs only a finite number of times and the process stops. □

From this proof, we obtain the following algorithm:

Exponents of solutions
Input:

- $k(t,\theta_{\mathcal{L}})$ where
  - $k(t)$ is a monomial extension of k and $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$,
  - $k(t,\theta_{\mathcal{L}})$ is a well-defined exponential extension of $k(t)$.
- $L(y) = D^n y + A_{n-1}D^{n-1} + \cdots + A_1 Dy + A_0 y$ is a LDO with A_i's in $k(t)$.

Output: A finite set $\mathcal{S}$ such that if for f in $k(t)$ and $(\beta_1,\ldots,\beta_l)$ in $\mathbb{Z}^l$ we have $L(f\theta_1^{\beta_1}\ldots\theta_l^{\beta_l}) = 0$, then $(\beta_1,\ldots,\beta_l)$ is in $\mathcal{S}$.

Algorithm:
Let $g_i = \frac{D\theta_i}{\theta_i}$,
and $\mathcal{P}^{\mathcal{L}} = \{p \in k[t]$ irreducible such that $\exists\, i \in \mathcal{L}$ such that $\mathrm{val}_p(g_i) < 0\}$.
If $l = 0$ then $\mathcal{S} = \emptyset$ else do
Let $\mathcal{Q}^{\mathcal{L}} = \{j \in \mathcal{L} \mid \deg_t(g_j) = \max(\deg_t(Dt), 1, \max_k(\deg_t(g_k)))\}$.
If $\mathcal{Q}^{\mathcal{L}} \neq \emptyset$ then
write $g_j = u_j t^\alpha + \cdots$ for j in $\mathcal{Q}^{\mathcal{L}}$ and $A_j = a_j t^{\alpha_j} + \cdots$,
find the integer solutions of $\sum_{k|k\alpha+\alpha_k=\max_h(\alpha_h+h\alpha)} a_k(\sum_{j\in\mathcal{Q}^{\mathcal{L}}}\beta_j u_j)^k = 0$.
If $\mathcal{Q}^{\mathcal{L}} = \emptyset$ then
let $\mathcal{Q}_p^{\mathcal{L}} = \{j \in \mathcal{L} \mid \mathrm{val}_p(g_j) = \min(-1, \min_k(\mathrm{val}_p(g_k)))\}$,
find a polynomial p in $\mathcal{P}^{\mathcal{L}}$ such that $\mathcal{Q}_p^{\mathcal{L}} \neq \emptyset$ and

if $g_j = \frac{u_j}{p^\alpha} + \cdots$ for $j \in \mathcal{Q}_p^{\mathcal{L}}$ then
the u_j's (and Dp mod p if $\alpha = 1$) are $\mathbb{Q}$-linearly independent.
Let $A_j = a_j p^{\alpha_j} + \cdots$.
If p is normal and $\alpha \geq 2$ or if p is special, compute the integer solutions of
$\sum_{k|k\alpha-\alpha_k=\max_h(h\alpha-\alpha_h)} a_k(\sum_{j\in\mathcal{Q}_p^{\mathcal{L}}} \beta_j u_j)^k \equiv 0 \bmod p$.
If p is normal and $\alpha = 1$, compute the integer solutions of
$\sum_{k|k-\alpha_k=\max_h(h-\alpha_h)} a_k \left(\prod_{j=0}^{k-1}\left(\left(\sum_{i\in\mathcal{Q}_p^{\mathcal{L}}} \beta_i u_i\right) - (j-m)Dp\right)\right) \equiv 0 \bmod p$.
For all the possibilities for $(\beta_i)_{i\in\mathcal{Q}^{\mathcal{L}}}$ or $(\beta_i)_{i\in\mathcal{Q}_p^{\mathcal{L}}}$ do
Let $Y = z\prod_{i\in\mathcal{Q}^{\mathcal{L}}} \theta_i^{\beta_i}$ or $Y = z\prod_{i\in\mathcal{Q}_p^{\mathcal{L}}} \theta_i^{\beta_i}$ in L and obtain $\widehat{L}$ such that $\widehat{L}(z) = 0$.
Apply the algorithm *exponents of solutions* to $k(t, \theta_{\mathcal{L}\setminus\mathcal{Q}^{\mathcal{L}}})$ or $k(t, \theta_{\mathcal{L}\setminus\mathcal{Q}_p^{\mathcal{L}}})$, and
$\widehat{L}(z) = 0$: compute a finite set $\widehat{\mathcal{S}}$ of possibilities for $(\beta_i)_{i\in\mathcal{L}\setminus\mathcal{Q}^{\mathcal{L}}}$ or $(\beta_i)_{i\in\mathcal{L}\setminus\mathcal{Q}_p^{\mathcal{L}}}$.
Let $\mathcal{S} = (\beta_i)_{i\in\mathcal{L}\setminus\mathcal{Q}^{\mathcal{L}}} \sqcup \widehat{\mathcal{S}}$ or $(\beta_i)_{i\in\mathcal{L}\setminus\mathcal{Q}_p^{\mathcal{L}}} \sqcup \widehat{\mathcal{S}}$.

Proof of correctness. As explained in the proof of the previous proposition, this algorithm stops because the cardinal of $\mathcal{L}$ strictly decreases, and gives us the possibilities for the exponents. Note that if $\mathcal{Q}^{\mathcal{L}} = \emptyset$, then, according to Definition 2.4, there exists a suitable polynomial p such that $\mathcal{Q}_p^{\mathcal{L}} \neq \emptyset$ and the leading terms of the p-adic expansions of the g_j for j in $\mathcal{Q}_p^{\mathcal{L}}$ are $\mathbb{Q}$-linearly independent. Furthermore, the finiteness of the number of integer solutions (β_i) is guaranteed by Lemma 2.5. $\square$

We can now compute bounds on the degree and the valuation of Laurent polynomial solutions of linear differential equations:

Theorem 2.7. *Let k be a differential field with a known $\mathbb{Q}$-basis and $k(t)$ be a monomial extension of k such that* $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$. *Let $K = k(t, \theta_{\mathcal{L}})$ be a well-defined exponential extension of $k(t)$. Let $L = \sum_{i=1}^n A_i D^i$ be a linear differential operator with coefficients in $k(t)[\theta_{\mathcal{L}}, \theta_{\mathcal{L}}^{-1}]$ and $b_1, \ldots, b_k$ be elements of $k(t)[\theta_{\mathcal{L}}, \theta_{\mathcal{L}}^{-1}]$.*

We can find $(m_1, \ldots, m_l)$ and $(M_1, \ldots, M_l)$ such that if for Y in $k(t)[\theta_{\mathcal{L}}, \theta_{\mathcal{L}}^{-1}]$ we have $L(Y) = \sum_j c_j b_j$ for some constant parameters c_j, then $\mathrm{val}_{\theta_i}(Y) \geq m_i$ *and* $\deg_{\theta_i}(Y) \leq M_i$.

Proof. We do the proof for M_1 and m_1. We choose an admissible order (see Cox et al. (1992)) on the exponential variables θ_i such that $\theta_1 > \theta_i$ for all $i \neq 1$ (for example the lexicographic order with $\theta_1 > \theta_2 > \cdots > \theta_l$). Suppose that $L(Y) = \sum c_j b_j$ where Y is in $k(t)[\theta_{\mathcal{L}}, \theta_{\mathcal{L}}^{-1}]$ and the c_j's are some constant parameters. We write

$$L = \sum_{(j_1,\ldots,j_l)=\nu}^{\mu} \theta_1^{j_1}\ldots\theta_l^{j_l} L_j \qquad \text{where } \begin{cases} \nu \text{ and } \mu \text{ in } \mathbb{Z}^l, \\ \nu \leq \mu \text{ with respect to the order,} \\ L_i \text{ is in } k(t)[D],\ L_\nu \neq 0 \text{ and } L_\mu \neq 0 \end{cases}$$

and

$$Y = \sum_{(i_1,\ldots,i_l)=\delta}^{\gamma} y_i\theta_1^{i_1}\ldots\theta_l^{i_l} \qquad \text{where } \begin{cases} \delta \text{ and } \gamma \text{ in } \mathbb{Z}^l, \\ \delta \leq \gamma \text{ with respect to the order,} \\ y_i \text{ is in } k(t),\ y_\delta \neq 0 \text{ and } y_\gamma \neq 0. \end{cases}$$

We have

$$L(Y) = \theta_1^{\nu_1} \dots \theta_l^{\nu_l} L_\nu(y_\delta \theta_1^{\delta_1} \dots \theta_l^{\delta_l}) + \dots + \theta_1^{\mu_1} \dots \theta_l^{\mu_l} L_\mu(y_\gamma \theta_1^{\gamma_1} \dots \theta_l^{\gamma_l}).$$

If $L(Y) = \sum c_j b_j$, then

- either $\nu + \delta$, the valuation of $L(Y)$, is greater than or equal to the valuation of the b_j's—in this case we get a lower bound for δ, with respect to the order we have chosen—or $L_\nu(y_\delta \theta_1^{\delta_1} \dots \theta_l^{\delta_l}) = 0$ and by Proposition 2.6, we get a finite set $\mathcal{S}_m$ such that $(\delta_1, \dots, \delta_l)$ is in $\mathcal{S}_m$. Then $m_1 = \min\{\delta_1 \mid (\delta_1, \dots, \delta_l) \in \mathcal{S}_m\}$.
- Either $\mu + \gamma$, the degree of $L(Y)$, is lower than or equal to the degree of the b_j's—in this case we get an upper bound for γ—or $L_\mu(y_\gamma \theta_1^{\gamma_1} \dots \theta_l^{\gamma_l}) = 0$ and by Proposition 2.6, we then get a finite set $\mathcal{S}_M$ such that $(\gamma_1, \dots, \gamma_l)$ is in $\mathcal{S}_M$. Then $M_1 = \max\{\gamma_1 \mid (\gamma_1, \dots, \gamma_l) \in \mathcal{S}_M\}$.

We change the order to compute the other bounds m_i and M_i. $\square$

2.4. Computing the coefficients

In order to compute the coefficients, we proceed by induction on l. The induction hypothesis is that we can effectively solve parametrized linear differential equations over $k(t)[\theta_{\mathcal{L}\setminus\{1\}}, \theta_{\mathcal{L}\setminus\{1\}}^{-1}]$ (see Definition 2.1). By hypothesis, we can do that over $k(t)$.

Let $L = \sum_{i=1}^n A_i D^i$ be a linear differential operator with coefficients in $k(t)[\theta_{\mathcal{L}}, \theta_{\mathcal{L}}^{-1}]$ and let $b_1, \dots, b_k$ be elements of $k(t)[\theta_{\mathcal{L}}, \theta_{\mathcal{L}}^{-1}]$.

Using the previous theorem, we compute bounds M_1 for the degree and m_1 for the valuation of Laurent polynomial solutions with respect to θ_1. Up to a change of variables, we can assume that $m_1 = 0$ and also that $\mathrm{val}_{\theta_1}(L) = 0$. Let

$$L = \sum_{k=0}^{\mu} \theta_1^k L_k \qquad \text{with } L_k \text{ in } k(t)[\theta_{\mathcal{L}\setminus\{1\}}, \theta_{\mathcal{L}\setminus\{1\}}^{-1}[D].$$

Suppose that there exist Y in $k(t)[\theta_{\mathcal{L}}, \theta_{\mathcal{L}}^{-1}]$ and constant parameters c_j such that $L(Y) = \sum c_j b_j$. If $M_1 = 0$ then Y is in $k(t)[\theta_{\mathcal{L}\setminus\{1\}}, \theta_{\mathcal{L}\setminus\{1\}}^{-1}]$. In this case,

$$\sum_{k=0}^{\mu} \theta_1^k L_k(Y) = \sum c_j b_j = \sum_{k=\min(\mathrm{val}_{\theta_1}(b_j))}^{\max(\deg_{\theta_1}(b_j))} \theta_1^k Q_k(c_j),$$

where the polynomials Q_k have coefficients in $k(t)[\theta_{\mathcal{L}\setminus\{1\}} \theta_{\mathcal{L}\setminus\{1\}}^{-1}]$. Equating the coefficients in θ_1 on both sides, we have conditions over the c_j's or linear differential equations with coefficients in $k(t)[\theta_{\mathcal{L}\setminus\{1\}}, \theta_{\mathcal{L}\setminus\{1\}}^{-1}]$ that can be solved by hypothesis. If $M_1 > 0$, we decompose Y: $Y = y_0 + \theta_1 q$ where $y_0 = Y_{|\theta_1=0}$ is in $k(t)[\theta_{\mathcal{L}\setminus\{1\}}, \theta_{\mathcal{L}\setminus\{1\}}^{-1}]$ and q is in $k(t)[\theta_{\mathcal{L}}, \theta_{\mathcal{L}}^{-1}]$ with $\deg_{\theta_1}(q) < M_1$. We have

$$L(Y) = L(y_0 + \theta_1 q) = L_0(y_0) + (L - L_0)(y_0) + L(\theta_1 q).$$

By the induction hypothesis, we can solve $L_0(y_0) = \sum c_j b_j(0)$, where $b_j(0)$ is defined as $b_{j|\theta_1=0}$. So we have $f_1, \ldots, f_r$ in $k(t)[\theta_{\mathcal{L}\setminus\{1\}}, \theta^{-1}_{\mathcal{L}\setminus\{1\}}]$ and a matrix $\mathcal{M}$ with constant coefficients such that y_0 is a solution if and only if

$$y_0 = \sum_{i=1}^{r} d_i f_i \qquad \text{and} \qquad \mathcal{M}(d_1, \ldots, d_r, c_1, \ldots, c_h)^{\mathrm{T}} = 0.$$

If we change Y into $\sum d_i f_i + \theta_1 q$ in the previous equation we conclude that there exists a linear differential operator $\hat{L}$ with coefficients in $k(t)[\theta_{\mathcal{L}}, \theta_{\mathcal{L}}^{-1}][D]$ such that

$$\begin{aligned}\sum c_j b_j &= L(Y) = L(y_0) + L(\theta_1 q),\\ &= \sum c_j b_j(0) + (L - L_0)\left(\sum d_i f_i\right) + \theta_1 \hat{L}(q).\end{aligned}$$

Then it follows that

$$\hat{L}(q) = \sum c_j \frac{b_j - b_j(0)}{\theta_1} - \sum d_i \frac{1}{\theta_1}(L - L_0) f_i.$$

Finally we solve this equation with $M - 1$ as the bound for the degree of q in θ_1.

Remark 6. This method was introduced in Bronstein and Fredet (1999) and generalized to a larger class of coefficient fields and functional equations in Bronstein (2000).

Link with the recurrence equation

This method is analogous to the one presented in Abramov et al. (1995) where a recurrence is used to compute the coefficients of polynomial solutions of a linear differential equation with coefficients in $C(x)$. Let us consider

$$L = \sum_{k=0}^{\mu} \theta_1^k L_k \qquad \text{with } L_k \text{ in } k(t)[\theta_{\mathcal{L}\setminus\{1\}}, \theta^{-1}_{\mathcal{L}\setminus\{1\}}][D].$$

Let f be an element of $k(t)[\theta_{\mathcal{L}\setminus\{1\}}, \theta^{-1}_{\mathcal{L}\setminus\{1\}}]$ and γ_1 be in $\mathbb{Z}$. We have

$$D^i(f\theta_1^{\gamma_1}) = \theta_1^{\gamma_1}(D + \gamma_1 g_1)^i(f) \qquad \text{for all } i \text{ in } \mathbb{N},$$

where g_1 is the logarithmic derivative of θ_1: $g_1 = \frac{D\theta_1}{\theta_1}$. Consequently, for any operator L_k in $k(t)[\theta_{\mathcal{L}\setminus\{1\}}, \theta^{-1}_{\mathcal{L}\setminus\{1\}}]$, we have

$$L_k(f\theta_1^{\gamma_1}) = \theta_1^{\gamma_1} S_{D\to D+\gamma_1 g_1}(L_k)(f),$$

where $S_{D\to D+\gamma_1 g_1}(L_k)$ is the resulting operator obtained from the operator L_k if D is replaced by $D + \gamma_1 g_1$. Then the following holds:

$$L(f\theta_1^{\gamma_1}) = \sum_{k=0}^{\mu} \theta_1^k L_k(f\theta_1^{\gamma_1}) = \sum_{k=0}^{\mu} \theta_1^{k+\gamma_1} S_{D\to D+\gamma_1 g_1}(L_k)(f).$$

Assuming a polynomial solution of the form $Y = \sum_{i=0}^{M_1} \theta_1^i Y_i$ (possibly up to a change of variable), we have

$$\begin{aligned} L(Y) &= \sum_{k=0}^{\mu} \sum_{i=0}^{M_1} \theta_1^{i+k} S_{D \to D+ig_1}(L_k)(y_i) \\ &= \sum_{j=0}^{\mu+M_1} \theta_1^j \sum_{k=0}^{\mu} S_{D \to D+(j-k)g_1}(L_k)(y_{j-k}). \end{aligned}$$

The equality $L(y) = \sum \theta_1^h \widehat{b_h}$ gives us a differential recurrence relation over the coefficients.

This is consistent with the results used for computing the bounds: if $j = k = 0$ then $L_0(y_0) = b_0$ and if $j = \mu + M_1$ and $k = \mu$ then $S_{D \to D+M_1 g_1}(L_\mu)(y_{M_1}) = b_{\mu+M_1}$.

The operator considered for the i-th step of the iteration in the specialization is $S_{D \to D+ig_1} L_0$. The important difference from the recurrence presented in Abramov et al. (1995) is that the coefficients of the recurrence are differential operators.

Improvements

There are several ways to improve the computation of the coefficients. The variable we consider first can be chosen or we can do the recurrence on several variables simultaneously. Let us sketch several cases of improvements:

1. Using asymptotics.

   Assuming that the extension is well defined, the logarithmic derivatives of the exponential variables have different orders or the same order and with $\mathbb{Q}$-linearly independent leading terms, for some asymptotic scale in the sense of Richardson et al. (1996). When computing bounds for the degree and the valuation of polynomial solutions, using the algorithm, we obtain first the exponents according to the leading logarithmic derivatives. If we consider the solutions as polynomial in this variables first, it is not necessary to iterate the process in the algorithm in order to compute the other exponents.

   Example 4. Let us consider the Laurent polynomial solutions of linear differential equation with coefficients in $C(x, \exp(x), \exp(x^2))$. In order to bound the degree and the valuation, one considers linear differential equations with coefficients in $C(x)$ and solutions of the form $f \exp(\gamma_1 x) \exp(\gamma_2 x^2)$ with f in $C(x)$ and γ_1, γ_2 in $\mathbb{Z}$. Using the algorithm given in the proof of Proposition 2.6, the possibilities for γ_2 will be computed first. So, it may be preferable to consider the solution as a Laurent polynomial in $\exp(x^2)$, with coefficients in $C[x, \exp(x), \exp(x)^{-1}]$.

2. Choosing the variable for the iteration.

   Using different orders to bound the degree and the valuation of Laurent polynomial solutions in several variables θ_i, we have a choice for the main variable to consider in order to compute the coefficients (for example, the variable with the smallest difference between the degree and the valuation).

Example 5. Let us consider the field $C(x, \exp(x^2), \exp(\sqrt{2}x^2)) = C(x, \theta_1, \theta_2)$ with the usual derivation. Let

$$\begin{aligned}L = {} & ((6\sqrt{2}x - 10x)\theta_2^3\theta_1^7 + (1 - 12x^2)\theta_1^6 + (2x + 4\sqrt{2}x)\theta_2^4\theta_1 \\ & + (1 - 2x^2\sqrt{2})\theta_2)D^2 \\ & + ((10 - 24x^2\sqrt{2} + 68x^2 - 6\sqrt{2})\theta_2^3\theta_1^7 + (12x + 144x^3)\theta_1^6 \\ & + (-68x^2 - 4\sqrt{2} - 24x^2\sqrt{2} - 2)\theta_2^4\theta_1 + (2\sqrt{2}x + 8x^3)\theta_2)D \\ & + (-576x^3\sqrt{2} + 624x^3)\theta_2^3\theta_1^7 + (-12 - 144x^2)\theta_1^6 \\ & + (80x^3 + 104x^3\sqrt{2})\theta_2^4\theta_1 + (-2\sqrt{2} - 8x^2)\theta_2.\end{aligned}$$

- Considering the lexicographic order with $\theta_1 > \theta_2$, we have

$$\begin{aligned}L = {} & ((6\sqrt{2} - 10)x\theta_1^7\theta_2^3 + \cdots + (1 - 2\sqrt{2}x^2)\theta_2)D^2 \\ & + (((-24\sqrt{2} + 68)x^2 + 10 - 6\sqrt{2})\theta_1^7\theta_2^3 + \cdots + (2\sqrt{2}x \\ & + 8x^3)\theta_2)D + (-576\sqrt{2} + 624)x^3\theta_1^7\theta_2^3 + \cdots + (-8x^2 - 2\sqrt{2})\theta_2.\end{aligned}$$

  – In order to bound the degree of a polynomial solution in $C(x, \theta_1, \theta_2)$, we have to solve the following equation:

$$\begin{aligned}& (-576\sqrt{2} + 624) + (-24\sqrt{2} + 68) \times 2(\gamma_1 + \sqrt{2}\gamma_2) \\ & \quad + (6\sqrt{2} - 10) \times 4(\gamma_1 + \sqrt{2}\gamma_2)^2 = 0.\end{aligned}$$

  Hence we get that the degree (γ_1, γ_2) must belong to $\{(6, 0), (1, 3)\}$. Therefore 6 is a bound for the degree in θ_1 of Laurent polynomial solutions and 0 is a bound for the degree in θ_2 of the leading term in θ_1.

  – For the valuation, we have to solve the following equation:

$$0 + 8 \times 2(\delta_1 + \sqrt{2}\delta_2) - 2\sqrt{2} \times 4(\delta_1 + \sqrt{2}\delta_2)^2 = 0.$$

  Thus we conclude that the valuation (δ_1, δ_2) is in $\{(0, 1), (0, 0)\}$.

- Considering the lexicographic order with $\theta_1 < \theta_2$, we have

$$\begin{aligned}L = {} & ((4\sqrt{2} + 2)x\theta_1\theta_2^4 + \cdots + (1 - 12x^2)\theta_1^6)D^2 \\ & + (((-68 - 24\sqrt{2})x^2 - 4\sqrt{2} - 2)\theta_1\theta_2^4 + \cdots + (12x \\ & + 144x^3)\theta_1^6)D + (104\sqrt{2} + 80)x^3\theta_1\theta_2^4 + \cdots + (-12 - 144x^2)\theta_1^6.\end{aligned}$$

  – In order to bound the degree of a polynomial solution $C(x, \theta_1, \theta_2)$, we have to solve the following equation:

$$\begin{aligned}& (104\sqrt{2} + 80) + (-68 - 24\sqrt{2}) \times 2(\gamma_1 + \sqrt{2}\gamma_2) \\ & \quad + (4\sqrt{2} + 2) \times 4(\gamma_1 + \sqrt{2}\gamma_2)^2 = 0.\end{aligned}$$

  Hence we conclude that (γ_1, γ_2) must belong to $\{(0, 1), (1, 3)\}$.

– For the valuation, we have to solve the following equation:

$$0 + 144 \times 2(\delta_1 + \sqrt{2}\delta_2) - 12 \times 4(\delta_1 + \sqrt{2}\delta_2)^2 = 0.$$

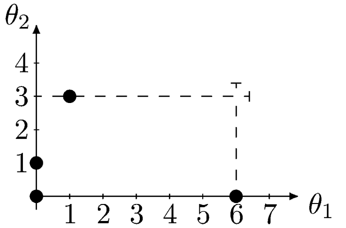
Thus, we conclude that (δ_1, δ_2) must be in $\{(6, 0), (0, 0)\}$.

If $L(Y) = 0$ for Y in $C(x)[\theta_1, \theta_2, \theta_1^{-1}, \theta_2^{-1}]$ then $\deg_{\theta_1}(Y) \leq 6$, $\mathrm{val}_{\theta_1}(Y) \geq 0$ and $\deg_{\theta_2}(Y) \leq 3$, $\mathrm{val}_{\theta_2}(Y) \geq 0$. So, by considering first the recurrence with θ_2, we have four linear differential equations with coefficients in $C(x)[\theta_1, \theta_1^{-1}]$ for computing the coefficients of Y, instead of the seven linear differential equations in $C(x)[\theta_2, \theta_2^{-1}]$ that we would have if we had started with θ_1.

3. Using several variables.

We can compute the bounds on several variables and use a recurrence on these variables simultaneously.

Example 6. Let us consider the field $C(x, \exp(x^2), \exp(\sqrt{2}x^2)) = C(x, \theta_1, \theta_2)$ and continue with Example 5. We have just shown that using the lexicographic order with $\theta_1 > \theta_2$, the extremal points are $(6, 0)$ for the degree, $(0, 0)$ for the valuation, and using the lexicographic order with $\theta_2 > \theta_1$, the extremal points are $(1, 3)$ for the degree and $(0, 0)$ for the valuation. Plotting these points in a graph, and considering the box they define, we have a finite set of possibilities for the exponents of θ_1 and θ_2.



So, the monomials in the solution have the form $\theta_1^{\beta_1}\theta_2^{\beta_2}$ where $0 \leq \beta_1 \leq 6$ and $0 \leq \beta_2 \leq 3$, i.e., $Y = \sum_{i=0}^{6}\sum_{j=0}^{3} y_{i,j}\theta_1^i\theta_2^j$. In fact, one can state precisely that $Y = y_{6,0}\theta_1^6 + \sum_{i=0}^{5}\sum_{j=0}^{3} y_{i,j}\theta_1^i\theta_2^j$. Plugging Y into the equation leads us to a linear differential system for the $y_{i,j}$'s with coefficients in $C(x)$.

Summarizing the previous results, we have the following theorem:

Theorem 2.8. *Let $k(t)$ be a monomial extension of k and* $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$ *such that one can effectively solve parametrized linear differential equations over $k(t)$. We can effectively solve parametrized linear differential equations over $k(t, \theta_{\mathcal{L}})$, where $k(t, \theta_{\mathcal{L}})$ is a well-defined exponential extension of $k(t)$.*

Proof. Follows from Theorem 2.7 and Section 2.4. □

2.5. Examples

2.5.1. Exponential extension of a monomial extension

Consider the extension

$$\begin{array}{cl} C(x,t,\theta) & \text{where } \theta \text{ is transcendental over } C(x,t) \text{ such that } \frac{D\theta}{\theta} = t^3, \\ | & \\ C(x,t) & \text{where } t \text{ is transcendental over } C(x) \text{ such that } Dt = t^2+1, \\ | & \\ C(x) & \text{where } Dx = 1. \end{array}$$

Let

$$\begin{aligned} L = {} & (\mathbf{2xt^6} + xt^4 - t^3 + xt^2)D^2 \\ & + (-\mathbf{8xt^9} - 16xt^7 + 2t^6 - 20xt^5 + 2t^4 - 6xt^3 + 2t^2 - 2xt)D \\ & + \mathbf{6xt^{12}} + 13xt^{10} + 3t^9 + 24xt^8 + 11t^7 + 26xt^6 \\ & + 13t^5 + 19xt^4 + 6xt^2 - 2t + 2x. \end{aligned} \tag{1}$$

We search for solutions of $L(y) = 0$ in $C(x)[t, \theta, \theta^{-1}]$. From Rosenlicht (1975), we know that if such a solution exists it can be searched for in the form $f\theta^\gamma$ where f is in $C(x)[t]$ and γ in $\mathbb{Z}$. We have

$$\begin{aligned} Y &= f\theta^\gamma \text{ with } f \text{ in } C(x)[t] \qquad \text{and} \qquad \gamma \text{ in } \mathbb{Z}, \\ DY &= (\gamma t^3 f + Df)\theta^\gamma, \\ D^2Y &= (\gamma^2 t^6 f + \cdots)\theta^\gamma. \end{aligned}$$

So $Q(\gamma) = 0$ where $Q = 2Z^2 - 8Z + 6 = 2(Z-1)(Z-3)$.

- Computing the solutions
  - If $\gamma = 1$:
    A change of variables gives the equation

    $$\begin{aligned} L_1 = {} & (2xt^6 + xt^4 - t^3 + xt^2)D^2 \\ & + (-\mathbf{4xt^9} - 14xt^7 - 18xt^5 + 2t^4 - 6xt^3 + 2t^2 - 2xt)D + \mathbf{4xt^{10}} \\ & + 4t^9 + 14xt^8 + 10t^7 + 26xt^6 + 12t^5 + 20xt^4 + 6xt^2 - 2t + 2x. \end{aligned}$$

    We have

    $$\begin{aligned} Y &= y_\gamma t^\gamma + \cdots \text{with } \gamma \text{ in } \mathbb{Z} \qquad \text{and} \qquad y_\gamma \text{ in } C(x), \\ DY &= \gamma y_\gamma t^{\gamma+1} + \cdots, \\ D^2Y &= \gamma(\gamma+1)t^{\gamma+2} + \cdots. \end{aligned}$$

    So $Q(\gamma) = 0$ where $Q = -4Z + 4 = 4(Z-1)$. A change of variable provides us with the solutions of $L_1(Y) = 0$ of the form c_1xt where c_1 is an arbitrary constant.
  - If $\gamma = 3$:
    A change of variables gives us the equation

$$\begin{aligned} L_2 = {} & (2xt^6 + xt^4 - t^3 + xt^2)D^2 \\ & + (\mathbf{4xt^9} - 10xt^7 - 4t^6 - 14xt^5 + 2t^4 - 6xt^3 + 2t^2 - 2xt)D \\ & - \mathbf{8xt^{10}} + 8t^7 + 26xt^6 + 10t^5 + 22xt^4 + 6xt^2 - 2t + 2x. \end{aligned}$$

So $Q(\gamma) = 0$ where $Q = 4Z - 8 = 4(Z - 2)$. A change of variable provides us with the solutions of $L_2(Y) = 0$ of the form c_2t^2 where c_2 is an arbitrary constant.

Solutions of $L(Y) = 0$ where L is defined in (1) are $c_1xt\theta + c_2t^2\theta^3$ where c_1 and c_2 are arbitrary constants.

Note that this extension is Liouvillian, equivalent to $C(x, T)$ where $T = \frac{t-i}{t+i}$ is such that $DT = 2iT$.

2.5.2. Extensions with logarithms

Algorithms concerning polynomial solutions of linear differential equations with coefficients in extensions generated by iterated logarithms and exponential were presented in Fredet (2000). We recall one example here, because it shows how a change of the derivation can simplify the computation of polynomial solutions.

Let us consider the Example 3.9.1 of Singer (1991): we are interested in the polynomial solutions of

$$L(y) := (x^2 \ln^2 x)y'' + (x \ln^2 x - 3x \ln x)y' + 3y = 0. \tag{2}$$

We consider $C[l_0, l_1] = \mathbb{Q}[x, \ln x]$ with the derivation D such that $Dl_1 = l_1$, $Dl_0 = l_0l_1$, i.e., $D = (x \ln x)\frac{\mathrm{d}}{\mathrm{d}x}$. So, we consider

$$L(y) = D^2 - 4D + 3$$

and we search for Y in $C[l_0, l_1, l_0^{-1}, l_1^{-1}]$ such that $L(Y) = 0$.

- Bounds on the degree and the valuation with respect to l_0:

  A solution of $L(Y) = 0$ is monomial in l_0 (because l_0 does not appear in the equation—see Rosenlicht (1975)). Let $Y = f l_0^\gamma$, where γ is in $\mathbb{Z}$ and f in $C[l_1, l_1^{-1}]$. So

  $$DY = (Df + \gamma l_1)l_0^\gamma \qquad \text{and} \qquad D^2Y = (D^2 f + \gamma l_1 + \gamma l_1 Df + \gamma^2 l_1^2)l_0^\gamma.$$

  Considering the leading term of $L(Y)$ with respect to l_1, we have $\gamma^2 = 0$, i.e., $\gamma = 0$. Then the polynomial solutions of $L(y) = 0$ are in $C[l_1, l_1^{-1}]$.
- Coefficients with respect to l_0:

  A solution of $L(Y) = 0$ is monomial in l_1 (because l_1 does not appear in the equation). Let $Y = fl_1^\gamma$, where γ is in $\mathbb{Z}$ and f in C. Then $DY = \gamma Y$, $D^2Y = \gamma^2Y$. So, we search for γ such that

  $$\gamma^2 - 4\gamma + 3 = 0 = (\gamma - 1)(\gamma - 3).$$

Solutions of (2) are $c_1l_1 + c_2l_1^3 = c_1 \ln x + c_2(\ln x)^3$ where c_1, c_2 are constant parameters.

3. Rewriting of the exponential extension

Given a flat effectively exponential extension of a monomial extension, we want to find a system of generators such that the extension is well defined.

3.1. Algorithm

We need the following lemmas:

Lemma 3.1. *Let $E = k(t, \theta_{\mathcal{L}})$ be a flat effectively exponential extension over $k(t)$ and consider*

$$\begin{aligned}
&\mathcal{Q}^{\max} = \{j \in \mathcal{L} \textit{ s.t. } \deg_t(g_j) \geq \max(\deg_t(Dt), 1)\},\\
&\mathcal{P} = \{\textit{irreducible } p \in k[t] \textit{ s.t. } \exists\, j \in \mathcal{L} \textit{ such that } \mathrm{val}_p(g_j) < 0\} \textit{ and}\\
&\mathcal{P}_{-1} = \{\textit{irreducible } p \in k[t] \textit{ s.t. } \exists\, j \in \mathcal{L} \textit{ s.t. } \mathrm{val}_p(g_j) = -1 \textit{ and if}\\
&\qquad g_j = \frac{u_j}{p} + \cdots \textit{ then the } u_j'\textit{s and } (Dp \bmod p) \textit{ are}\\
&\qquad \mathbb{Q}\textit{-linearly dependent}\}.
\end{aligned}$$

If $\mathcal{Q}^{\max} = \emptyset$, then $\mathcal{P} \backslash \mathcal{P}_{-1} \neq \emptyset$.

Proof. According to Definition 2.2, if $\mathcal{Q}^{\max} = \emptyset$, then for all i in $\{1, \ldots, l\}$, for all c in $\mathbb{Q}$ and f in $k(t)^*$, $\frac{D\theta_i}{\theta_i} + c\frac{Df}{f}$ has a $\frac{1}{t}$-adic expansion containing an element with the form $u_\beta t^\beta$ where $\beta < 0$ is an integer. This means that there exists an irreducible polynomial $p \in k[t]$ such that $\frac{D\theta_i}{\theta_i} + c\frac{Df}{f}$ has a p-adic expansion with the form $\frac{u_i}{p^\alpha} + \cdots$, with $\alpha > 0$. With $c = 0$ this implies that $\mathcal{P} \neq \emptyset$. Furthermore, for each exponential variable θ, as already remarked after Definition 2.2, there exists $p \in k[t]$ such that either $\mathrm{val}_p(\frac{D\theta}{\theta}) < -1$, or $\mathrm{val}_p(\frac{D\theta}{\theta}) = -1$ and the leading term in the p-adic expansion is $\mathbb{Q}$-linearly independent of $(Dp \bmod p)$. So this implies that, for each exponential variable, there exists $p \in \mathcal{P} \backslash \mathcal{P}^{-1}$ and then $\mathcal{P} \backslash \mathcal{P}_{-1} \neq \emptyset$. $\square$

Lemma 3.2. *Let $E = k(t, \theta_{\mathcal{L}})$ be a flat effectively exponential extension over $k(t)$. Let $\mathcal{B}$ and $\mathcal{Q}$ be two subsets of $\mathcal{L}$ such that $\mathcal{B} \cap \mathcal{Q} = \emptyset$. Define $\widehat{\theta}_{\mathcal{L}}$ such that*

- *If i is in $\mathcal{B}$, then $\widehat{\theta}_i = \theta_i^{1/d_i}$, i.e., $\frac{D\widehat{\theta}_i}{\widehat{\theta}_i} = \frac{1}{d_i}\frac{D\theta_i}{\theta_i}$ for some d_i in $\mathbb{Z}_{\neq 0}$.*
- *If i is in $\mathcal{Q}$, then $\widehat{\theta}_i = \theta_i/(\prod_{j\in\mathcal{B}} \theta_j^{c_{i,j}/d_j})$, i.e., $\frac{D\widehat{\theta}_i}{\widehat{\theta}_i} = \frac{D\theta_i}{\theta_i} - \sum_{j\in\mathcal{B}} \frac{c_{i,j}}{d_j}\frac{D\theta_j}{\theta_j}$ for some $c_{i,j}$ in $\mathbb{Z}$.*
- *If i is in $\mathcal{L}\backslash(\mathcal{B} \cup \mathcal{Q})$, then $\widehat{\theta}_i = \theta_i$.*

Then $k(t, \widehat{\theta}_{\mathcal{L}})$ is a flat effectively exponential extension over $k(t)$.

Proof. Let e_i be in $\mathbb{Q}$ and define $\mathcal{E} = \mathcal{L}\backslash(\mathcal{Q} \cup \mathcal{B})$. Then

$$\prod_{i\in\mathcal{L}} \widehat{\theta}_i^{e_i} = \prod_{i\in\mathcal{B}} \widehat{\theta}_i^{e_i} \prod_{i\in\mathcal{Q}} \widehat{\theta}_i^{e_i} \prod_{i\in\mathcal{E}} \widehat{\theta}_i^{e_i} = \prod_{i\in\mathcal{B}} \theta_i^{(e_i - \sum_{j\in\mathcal{Q}} c_{j,i}e_j)/d_i} \prod_{i\in\mathcal{Q}} \theta_i^{e_i} \prod_{i\in\mathcal{E}} \theta_i^{e_i}.$$

If $(e_i)_{i\in\mathcal{L}}$ are not all zero, this is also the case for $((e_i - \sum_{j\in\mathcal{Q}} c_{j,i}e_j)/d_i)_{i\in\mathcal{B}}$ and $(e_i)_{i\in\mathcal{Q}\cup\mathcal{E}}$ because if $((e_i - \sum_{j\in\mathcal{Q}} c_{j,i}e_j)/d_i)_{i\in\mathcal{B}}$ and $(e_i)_{i\in\mathcal{Q}}$ are all zero, then $(e_i)_{i\in\mathcal{B}}$ are also all zero. Then any product of rational powers, not all zero, of the $\widehat{\theta}_i$'s is a product of rational powers, not all zero, of the θ_i's. According to Definition 2.3, such a product of the θ_i's is effectively exponential. Then, any product of rational powers (not all zero) of the $\widehat{\theta_i}$'s is also effectively exponential over $k(t)$. This implies that $k(t, \widehat{\theta_\mathcal{L}})$ is a flat effectively exponential extension over $k(t)$. $\square$

Theorem 3.3. *Let $E = k(t, \theta_\mathcal{L})$ be a flat effectively exponential extension over $k(t)$. We can compute a set of generators $(\widehat{\theta_\mathcal{L}})$ such that $k(t, \widehat{\theta_\mathcal{L}})$ is an algebraic extension of $k(t, \theta_\mathcal{L})$ which is flat effectively exponential over $k(t)$ and well defined.*

Proof. We give an algorithm that computes such a set of generators. In order to do this, we introduce two sets $\mathcal{M}$ and $\mathcal{T}$ such that $\mathcal{M} = \mathcal{L} \backslash \mathcal{T}$. The induction hypothesis is that if $k(t, \theta_\mathcal{N})$ is not well defined for some minimal set $\mathcal{N} \subset \mathcal{L}$ then $\mathcal{N} \subset \mathcal{T}$, where the set $\mathcal{N}$ is *minimal* if for any subset $\mathcal{N}' \subsetneq \mathcal{N}$ the extension $k(t, \theta_{\mathcal{N}'})$ is well defined.

We start with $\mathcal{M} = \emptyset$ and $\mathcal{T} = \mathcal{L}$. The main idea is to find expansions such that the leading terms of the logarithmic derivatives of some elements in $\mathcal{T}$ are $\mathbb{Q}$-linearly independent, maybe up to an algebraic rewriting of the exponential elements. Then we will add the subscripts of these elements to $\mathcal{M}$ and subtract them from $\mathcal{T}$ until $\mathcal{T} = \emptyset$. At the end, we consider the extension $k(t, \theta_\mathcal{M})$ which is well defined.

At each step, we will define some sets $\mathcal{Q}$ and $\mathcal{B}$ and rewrite the exponential extension as in Lemma 3.2. So, according to this lemma, the new exponential extension will still be effectively exponential.

For $j \in \mathcal{T}$, we write $g_j = \frac{D\theta_j}{\theta_j}$ and, according to Definition 2.4, we consider

$$\mathcal{Q}^\mathcal{T} = \left\{ j \in \mathcal{T} \mid \deg_t(g_j) = \max\left(\deg_t(Dt), 1, \max_{k \in \mathcal{T}}(\deg_t(g_k))\right)\right\}.$$

There are several steps:

1. **Step 1.** $\mathcal{Q}^\mathcal{T} \neq \emptyset$.

   We assume, as an induction hypothesis, that the degree of the logarithmic derivative of the variables θ_j for j in $\mathcal{T}$ is strictly less than the degree of the logarithmic derivative of the exponential variable θ_j for j in $\mathcal{M}$.

   If for j in $\mathcal{Q}^\mathcal{T}$ we write $g_j = u_j t^\alpha + \cdots$, then

   (a) either the u_j's are $\mathbb{Q}$-linearly independent and we can use the expansion at infinity to distinguish the leading terms of these logarithmic derivatives—in this case, we add $\mathcal{Q}^\mathcal{T}$ to $\mathcal{M}$ and remove it from $\mathcal{T}$;

   (b) or the u_j's are $\mathbb{Q}$-linearly dependent—then we compute $\mathcal{B}$ such that $(u_j)_{j \in \mathcal{B}}$ is a $\mathbb{Q}$-basis of $(u_j)_{j \in \mathcal{Q}^\mathcal{T}}$. We compute d in $\mathbb{Z}$ and $c_{i,j}$ in $\mathbb{Z}$ such that for all i in $\mathcal{Q}^\mathcal{T} \backslash \mathcal{B}$, $du_i = \sum_{j \in \mathcal{B}} c_{i,j} u_j$. So, we define the new exponential variables $\widehat{\theta_i} = \theta_i^{1/d}$ for i in $\mathcal{B}$ and $\widehat{\theta_i} = \theta_i / (\prod_{j \in \mathcal{B}} \widehat{\theta_l}^{c_{i,j}})$ for i in $\mathcal{Q}^\mathcal{T} \backslash \mathcal{B}$.
   Remark that if $d > 1$, then we do an algebraic (radical) extension of the exponential extension $k(t, \theta_\mathcal{M})$.
   For j in $\mathcal{B}$, we include the new variables θ_j in the set of the well-defined variables, i.e., $\widehat{\mathcal{M}} = \mathcal{M} \cup \mathcal{B}$ and $\widehat{\mathcal{T}} = \mathcal{T} \backslash \mathcal{B}$.

   Note that, in both cases, the degree of the logarithmic derivative of the variables θ_j for j in $\widehat{\mathcal{T}}$ is strictly less than the degree of the logarithmic derivative of the exponential variables θ_j for j in $\widehat{\mathcal{M}}$, as assumed at the beginning of this step. We will now see that the extension $k(t, \theta_{\widehat{\mathcal{M}}})$ is well defined: for any subset $\mathcal{N} \subset \widehat{\mathcal{M}}$, we have

- If $\mathcal{N} \subset \mathcal{M}$, then the extension $k(t, \theta_{\mathcal{N}})$ is well defined by the induction hypothesis.
- If $\mathcal{N} \cap \mathcal{M} = \emptyset$, then $\mathcal{N}$ contains only the subscripts of some new exponential variables $\widehat{\theta_j}$. So the leading terms are $\mathbb{Q}$-linearly independent.
- If $\mathcal{N} \not\subset \mathcal{M}$ and $\mathcal{N} \cap \mathcal{M} \neq \emptyset$, then $\mathcal{N}$ contains subscripts of some new variables and some old ones. The logarithmic derivatives of the variables θ_j for j in $\mathcal{M}$ have degree strictly greater than α, the degree of the logarithmic derivative of the new exponential variables $\widehat{\theta_i}$. Furthermore, the leading terms of the logarithmic derivatives with higher degree are $\mathbb{Q}$-linearly independent, by the induction hypothesis.

So, the extension $k(t, \theta_{\widehat{\mathcal{M}}})$ is well defined. Note that, according to Lemma 3.2, the new exponential extension is a flat effectively exponential of $k(t)$. We iterate Step 1 until $\mathcal{Q}^{\mathcal{T}} = \emptyset$.

2. **Step 2.** $\mathcal{Q}^{\mathcal{T}} = \emptyset$ and $\mathcal{T} \neq \emptyset$.

   According to Definition 2.4, we note

$$\mathcal{P}^{\mathcal{T}} = \{\text{irreducible } p \in k[t] \mid \exists j \in \mathcal{T} \text{ such that } \mathrm{val}_p(g_j) < 0\}$$

and

$$\mathcal{Q}_p^{\mathcal{T}} = \left\{ j \in \mathcal{T} \text{ such that } \mathrm{val}_p(g_j) = \min\left(-1, \min_{k \in \mathcal{T}}(\mathrm{val}_p(g_k))\right)\right\}.$$

We consider $\mathcal{P}^{\mathcal{T}}$ as an ordered list: $\mathcal{P}^{\mathcal{T}} = [p_1, \dots, p_r]$.

Remark 4 shows that if $\mathcal{Q}_p^{\mathcal{T}} \neq \emptyset$, then p is in $\mathcal{P}^{\mathcal{T}}$. Conversely, by definition, if p is in $\mathcal{P}^{\mathcal{T}}$, then $\mathcal{Q}_p^{\mathcal{T}} \neq \emptyset$.

We consider the first polynomial $p = p_1$ of $\mathcal{P}^{\mathcal{T}}$. For j in $\mathcal{Q}_p^{\mathcal{T}}$, we write $g_j = \frac{u_j}{p^{\alpha_p}} + \cdots$. We have two possibilities:

(a) If $\alpha_p > 1$, then we proceed as in Step 1:

  (i) Either the u_j's are $\mathbb{Q}$-linearly independent and we can use this p-adic expansion to distinguish the leading terms of these logarithmic derivatives. So we add $\mathcal{Q}_p^{\mathcal{T}}$ to $\mathcal{M}$ and remove it from $\mathcal{T}$: $\widehat{\mathcal{M}} := \mathcal{M} \cup \mathcal{Q}_p^{\mathcal{T}}$ and $\widehat{\mathcal{T}} := \mathcal{T} \setminus \mathcal{Q}_p^{\mathcal{T}}$.

  (ii) Or the u_j's are $\mathbb{Q}$-linearly dependent. Then we rewrite the exponential variables θ_j for j in $\mathcal{Q}_p^{\mathcal{T}}$: we compute a set $\mathcal{B}$ such that $(u_j)_{j \in \mathcal{B}}$ is a $\mathbb{Q}$-basis of $(u_j)_{j \in \mathcal{Q}_p^{\mathcal{T}}}$. We find d in $\mathbb{Z}$ and $c_{i,j}$ in $\mathbb{Z}$ such that for all i in $\mathcal{Q}_p^{\mathcal{T}} \setminus \mathcal{B}$, $u_i = \frac{1}{d} \sum_{j \in \mathcal{B}} c_{i,j} u_j$. We define the new exponential variables: $\widehat{\theta_i} = \theta_i^{1/d}$ for i in $\mathcal{B}$ and $\widehat{\theta_i} = \theta_i / (\prod_{j \in \mathcal{B}} \widehat{\theta_l}^{c_{i,j}})$ for i in $\mathcal{Q}_p^{\mathcal{T}} \setminus \mathcal{B}$.

  As in Step 1(b), the new exponential variables $(\widehat{\theta_i})_{i \in \mathcal{B}}$ may be radical over $k(t, \theta_{\mathcal{L}})$.

For $j \in \mathcal{B}$, we include the new variables $\widehat{\theta_j}$ in the set of the well-defined variables, i.e., $\widehat{\mathcal{M}} = \mathcal{M} \cup \mathcal{B}$ and $\widehat{\mathcal{T}} = \mathcal{T} \backslash \mathcal{B}$.

Note that for $j \in \widehat{\mathcal{T}}$, the logarithmic derivative of the new variable $\widehat{\theta_j}$ has valuation in p strictly greater than α_p.
If there exists $j \in \widehat{\mathcal{T}}$ such that $\mathrm{val}_p(g_j) < 0$, then we iterate Step 2 with the same p; else we remove p from $\mathcal{P}^{\mathcal{T}}$ and iterate Step 2.

(b) If $\alpha_p = 1$, then

(i) Either the u_j's and $(Dp \bmod p)$ are $\mathbb{Q}$-linearly independent. Then we can use this p-adic expansion to distinguish the leading terms of these logarithmic derivatives. So we add $\mathcal{Q}_p^{\mathcal{T}}$ to $\mathcal{M}$ and remove it from $\mathcal{T}$.

(ii) Or the u_j's and $(Dp \bmod p)$ are $\mathbb{Q}$-linearly dependent.
Then are again two possibilities:

A. Either all the u_j's are all $\mathbb{Q}$-linearly dependent of $(Dp \bmod p)$—then we can remove p from the set $\mathcal{P}^{\mathcal{T}}$ of the polynomials to consider. As $\mathcal{P}^{\mathcal{T}}$ is finite, this happens only a finite number of times; furthermore Lemma 3.1 shows that $\mathcal{P}$ contains other polynomials if $\mathcal{T} \neq \emptyset$. We iterate Step 2 with the next polynomial p_2.

B. Or the subset $\mathcal{B} \subset \mathcal{Q}_p^{\mathcal{T}}$ such that $(Dp \bmod p, (u_j)_{j \in \mathcal{B}})$ is a $\mathbb{Q}$-basis of $(Dp \bmod p, (u_j)_{j \in \mathcal{Q}_p^{\mathcal{T}}})$ is not empty—then we find d in $\mathbb{Z}$, $c_{0,i}$ in $\mathbb{Q}$ and $c_{i,j}$ in $\mathbb{Z}$ such that for all i in $\mathcal{Q}_p^{\mathcal{T}} \backslash \mathcal{B}$, $u_i = c_{i,0}(Dp \bmod p) + \frac{1}{d} \sum_{j \in \mathcal{B}} c_{i,j} u_j$. We define the new exponential variables: $\widehat{\theta_i} = \theta_i^{1/d}$ for i in $\mathcal{B}$ and $\widehat{\theta_i} = \theta_i / (\prod_{j \in \mathcal{B}} \widehat{\theta_l}^{c_{i,j}})$ for i in $\mathcal{Q}_p^{\mathcal{T}} \backslash \mathcal{B}$.
Note that for j in $\mathcal{Q}_p^{\mathcal{T}} \backslash \mathcal{B}$ the leading term of the p-adic expansions of the logarithmic derivative of $\widehat{\theta_j}$ has valuation 0 or -1. Furthermore, if the valuation is -1 then the leading coefficient is $\mathbb{Q}$-linearly dependent of $Dp \bmod p$. We define $\widehat{\mathcal{M}} = \mathcal{M} \cup \mathcal{B}$ and $\widehat{\mathcal{T}} = \mathcal{T} \backslash \mathcal{B}$.

Part (b) will happen only once for each p. So we remove p from $\mathcal{P}^{\mathcal{T}}$: we consider $\mathcal{P}^{\widehat{\mathcal{T}}}$, with the same order as $\mathcal{P}^{\mathcal{T}}$, remarking that $\mathcal{P}^{\widehat{\mathcal{T}}} \subset \mathcal{P}^{\mathcal{T}}$. In all cases, we iterate Step 2 with another polynomial.

We will now see that the extension $k(t, \theta_{\widehat{\mathcal{M}}})$ is well defined: for any subset $\mathcal{N} \subset \widehat{\mathcal{M}}$, we have

- If $\mathcal{N} \subset \mathcal{M}$, the extension $k(t, \theta_{\mathcal{N}})$ is well defined by the induction hypothesis.
- If $\mathcal{N} \cap \mathcal{M} = \emptyset$, then $\mathcal{N}$ contains only the subscripts of some new exponential variables $\widehat{\theta_j}$. So the leading terms of the p-adic expansion, and $Dp \bmod p$ in the case of valuation -1, are $\mathbb{Q}$-linearly independent.
- If $\mathcal{N} \not\subset \mathcal{M}$ and $\mathcal{N} \cap \mathcal{M} \neq \emptyset$, then $\mathcal{N}$ contains subscripts of some new variables and some old ones. So:

- either the logarithmic derivatives of the variables $(\theta_j)_{j\in\mathcal{N}\cap\mathcal{M}}$ have valuation at p strictly smaller than α, the valuation of the logarithmic derivative of the new exponential variable $\widehat{\theta_j}$,
- or p is involved only in the logarithmic derivatives of the new exponential variables $\widehat{\theta_j}$,
- or there exists another polynomial $\widehat{p}$ such that $\mathrm{val}_{\widehat{p}}(g_j) < \mathrm{val}_{\widehat{p}}(g_i)$ for j in $\mathcal{M}$ and i in $\mathcal{B} = \widehat{\mathcal{M}}\backslash\mathcal{M}$,
- or $\mathcal{Q}^{\mathcal{M}} \neq \emptyset$.

In all cases, considering the expansions at infinity or a suitable p-adic expansions, the leading terms are $\mathbb{Q}$-linearly independent by the induction hypothesis.

So, the extension $k(t, \theta_{\widehat{\mathcal{M}}})$ is well defined. Note that, according to Lemma 3.2, the new exponential extension is a flat effectively exponential of $k(t)$.

We iterate Step 2 until $\mathcal{P}^{\widehat{\mathcal{T}}} = \emptyset$, which implies that $\widehat{\mathcal{T}} = \emptyset$ using Lemma 3.1. □

Let us now give the algorithm:

Rewriting of the extension
Input:

- $k(t, \theta_{\mathcal{L}})$ where
  - $k(t)$ is a monomial extension of k and $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$,
  - $k(t, \theta_{\mathcal{L}})$ is a flat effectively exponential extension of $k(t)$.

Output: $\widehat{\theta_{\mathcal{L}}}$ such that $k(t, \widehat{\theta_{\mathcal{L}}})$ is an algebraic extension of $k(t, \theta_{\mathcal{L}})$ and a well-defined exponential extension of $k(t)$.

Algorithm:
Define $\mathcal{T} = \{1, \cdots, l\}$, $\mathcal{M} = \emptyset$.
While $\mathcal{T} \neq \emptyset$ do
  Let $\mathcal{Q}^{\mathcal{T}} = \{j \in \mathcal{T} \mid \deg_t(g_j) = \max(\deg_t(Dt), 1, \max_k(\deg_t g_k))\}$.
  Step 1: While $\mathcal{Q}^{\mathcal{T}} \neq \emptyset$ do
    Let $\alpha = \max_j(\deg_t(g_j))$. Then $\mathcal{Q}^{\mathcal{T}} = \{j \in \mathcal{T} \mid \deg_t(g_j) = \alpha\}$.
    For $i \in \mathcal{Q}^{\mathcal{T}}$, write $g_i = u_i t^{\alpha} + \cdots$.
    Let $(u_i)_{i\in\mathcal{B}}$ be a basis of $(u_i)_{i\in\mathcal{Q}^{\mathcal{T}}}$.
    Compute $d \in \mathbb{Z}$ and $c_{i,j} \in \mathbb{Z}$ such that $\forall i \in \mathcal{Q}^{\mathcal{T}}\backslash\mathcal{B}$, $du_i = \sum_{j\in\mathcal{B}} c_{i,j} u_j$.

$$\mathcal{M} \leftarrow \mathcal{M} \cup \mathcal{B}, \mathcal{T} \leftarrow \mathcal{T}\backslash\mathcal{B},$$

$$\theta_i \leftarrow \theta_i / (\textstyle\prod_{j\in\mathcal{B}} \theta_l^{c_{i,j}/d}) \text{ for } i \text{ in } \mathcal{Q}^{\mathcal{T}}\backslash\mathcal{B}, \theta_i \leftarrow \theta_i^{1/d} \text{ for } i \text{ in } \mathcal{B}.$$

  Loop.
  Step 2: If $\mathcal{Q}^{\mathcal{T}} = \emptyset$ do
  Let $\mathcal{P}^{\mathcal{T}}$ = [list of irreducible $p \in k[t] \mid \exists j \in \mathcal{T}$ such that $\mathrm{val}_p(g_j) < 0$] = $[p, p_2, \ldots, p_r]$, for the first polynomial p in $\mathcal{P}^{\mathcal{T}}$ do
    Let $\mathcal{Q}_p^{\mathcal{T}} = \left\{j \in \mathcal{T} \mid \mathrm{val}_p(g_j) = \min\left(-1, \min_k(\mathrm{val}_p(g_k))\right)\right\}$.
    Let $\alpha_p = -\min_j(\mathrm{val}_p(g_j))$ and for i in $\mathcal{Q}_p^{\mathcal{T}}$, write $g_i = \frac{u_i}{p^{\alpha_p}} + \cdots$.

If $\alpha_p > 1$: let $(u_i)_{i\in\mathcal{B}}$ be a basis of $(u_i)_{i\in\mathcal{Q}_p^{\mathcal{T}}}$.
Compute $d \in \mathbb{Z}$ and $c_{i,j} \in \mathbb{Z}$ such that $\forall i \in \mathcal{Q}_p^{\mathcal{T}} \backslash \mathcal{B}$, $du_i = \sum_{j\in\mathcal{B}} c_{i,j} u_j$.
Let $\mathcal{M} \leftarrow \mathcal{M} \cup \mathcal{B}$, $\mathcal{T} \leftarrow \mathcal{T}\backslash\mathcal{B}$,
$\theta_i \leftarrow \theta_i/(\prod_{j\in\mathcal{B}} \theta_l^{c_{i,j}/d})$ for i in $\mathcal{Q}_p^{\mathcal{T}}\backslash\mathcal{B}$, $\theta_i \leftarrow \theta_i^{1/d}$ for i in $\mathcal{B}$.
If $\alpha_p = 1$: let $q = (Dp \bmod p)$ and $(q, (u_i)_{i\in\mathcal{B}})$ be a basis of $(q, (u_i)_{i\in\mathcal{Q}_p^{\mathcal{T}}})$.
If $\mathcal{B} = \emptyset$ then $\mathcal{P}^{\mathcal{T}} \leftarrow \mathcal{P}^{\mathcal{T}}\backslash\{p\}$.
If $\mathcal{B} \neq \emptyset$ then find $d \in \mathbb{Z}$, $c_{0,i} \in \mathbb{Q}$ and $c_{i,j} \in \mathbb{Z}$ such that
for all $i \in \mathcal{Q}_p^{\mathcal{T}}\backslash\mathcal{B}$, $u_i = c_{i,0}q + \frac{1}{d}\sum_{j\in\mathcal{B}} c_{i,j}u_j$.
Let $\mathcal{M} \leftarrow \mathcal{M} \cup \mathcal{B}$, $\mathcal{T} \leftarrow \mathcal{T}\backslash\mathcal{B}$,
$\theta_i \leftarrow \theta_i/(\prod_{j\in\mathcal{B}} \theta_l^{c_{i,j}/d})$ for i in $\mathcal{Q}_p^{\mathcal{T}}\backslash\mathcal{B}$, $\theta_i \leftarrow \theta_i^{1/d}$ for i in $\mathcal{B}$.
$\mathcal{P}^{\widehat{\mathcal{T}}}$ = [list of irreducible $p \in k[t] \mid \exists j \in \widehat{\mathcal{T}}$ such that $\mathrm{val}_p(g_j) < 0$].
Loop with $\mathcal{P}^{\widehat{\mathcal{T}}}$ ordered as $\mathcal{P}^{\mathcal{T}}$.
Loop.
Return $\theta_{\mathcal{L}}$.

Proof of correctness. The algorithm stops because the number of iterations in which $|\mathcal{M}|$ does not strictly increase is finite (and smaller than $|\mathcal{P}|$). Furthermore we do not add any new constant. At the end, $\mathcal{T} = \emptyset$ and $\mathcal{M} = \mathcal{L}$ which implies that the extension is well defined as explained in the proof of Theorem 3.3. □

3.2. *Improvements*

There are several ways to improve the previous algorithm. The key idea is to avoid as much as possible rewriting and algebraic extension.

- If the u_j's are pairwise linearly dependent over $\mathbb{Q}$, we can modify the set of generators without extending the extension. For example, if $l = 2$ and if $\frac{u_1}{u_2} = \frac{p}{q}$, then from the Bézout theorem, there exist a and b such that $ap+bq = 1$. In this case, we define $\widehat{g_1} = ag_1 + bg_2$, $\widehat{g_2} = -qg_1 + pg_2$ and $\widehat{\theta_1} = \exp(\int g_1)$, $\theta_2 = \exp(\int g_2)$ (the extension is isomorphic because $g_1 = p\widehat{g_1} + q\widehat{g_2}$, and $g_2 = -v\widehat{g_1} + u\widehat{g_2}$).

  Example 7. If we consider $C(x, e^{\int 2x^2+3x+1}, e^{\int 3x^2+x})$, the algorithm outputs $C(x, e^{\int x^2+\frac{3}{2}x+\frac{1}{2}}, e^{\int -\frac{1}{6}x-\frac{1}{2}})$ or $C(x, e^{\int x^2+\frac{1}{3}}, e^{\int \frac{1}{6}x+\frac{1}{2}})$, depending on the choice for the $\mathbb{Q}$-basis $\mathcal{B}$. Both are algebraic extensions of $C(x, e^{\int 2x^2+3x+1}, e^{\int 3x^2+x})$. But it is isomorphic to $C(x, e^{\int x^2-2x-1}, e^{\int 7x+3})$, which is well defined, and can be found using the Bézout theorem.

- The order we choose on $\mathcal{P}^{\mathcal{T}}$ is important: we could start with polynomials p such that the cardinal of $\mathcal{Q}_p^{\mathcal{T}}\backslash\mathcal{B}$ is as small as possible, which leads to the least number of rewritings.

  Example 8. Consider $C(x, \theta_1, \theta_2)$ with $\frac{D\theta_1}{\theta_1} = \frac{2}{(x+1)^2} + \frac{1}{(x+3)^3}$ and $\frac{D\theta_2}{\theta_2} = \frac{3}{(x+1)^3} + \frac{1}{(x+2)^3}$. If we consider $\mathcal{P}^{\{1,2\}} = [x+1, x+2, x+3]$ or $[x+1,$

$x+3, x+2]$, depending on the choice for the $\mathbb{Q}$-basis $\mathcal{B}$, the algorithm outputs $C(x, e^{\int \frac{1}{(x+1)^2}+\frac{1/2}{(x+3)^3}}, e^{\int \frac{1}{(x+2)^3}-\frac{3/2}{(x+3)^3}})$, or $C(x, e^{\int \frac{2}{(x+1)^2}-\frac{2/3}{(x+2)^3}}, e^{\int \frac{1}{(x+1)^3}+\frac{1/3}{(x+2)^3}})$. Both are algebraic extensions of $C(x, \theta_1, \theta_2)$. But if we consider $\mathcal{P}^{\{1,2\}} = [x+2, x+3, x+1], [x+2, x+1, x+3], [x+3, x+2, x+1], [x+3, x+1, x+2]$, the algorithm remarks that this extension is already well defined.

4. Conclusion

We have presented algorithms for computing Laurent polynomial solutions of linear differential equations with coefficients in exponential extensions of monomial extensions of a base field. Our claim is that a suitable system of generators of the extension improves the computation of such solutions. In Fredet (2001), the suitable form of the system of generators has also been applied to compute Laurent polynomial solutions of linear differential systems with coefficients in well-defined exponential extensions. It should appear in subsequent works.

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